# RUDIMENTS

OF

# PLANE GEOMETRY,

INCLUDING

# GEOMETRICAL ANALYSIS,

AND

# PLANE TRIGONOMETRY.

*DESIGNED CHIEFLY FOR PROFESSIONAL MEN.*

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BY

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# PREFACE.

When I occupied the Mathematical Chair, I was led to consider closely the difficulties and impediments in the ordinary mode of teaching Geometry. I therefore resolved to frame a system of instruction, well balanced, and suitable to the advanced state of science. My great object was to unite theory with practice, to connect the ancient with the modern discoveries, and to avoid the prolixity, without departing from the strictness and beauty, of Greek demonstration.

Though part only of the original plan has been executed, it yet includes a tolerably complete body of Elementary Geometry. This digest, however, was designed for Students who engage in a course of six months, and require their powers of reasoning to be exer-

cised by a slow and rather extended procedure. But many individuals of riper minds are anxious to supply the defects of their early education, and eager to obtain, by a moderate share of application, such an acquaintance with Mathematics, as will qualify them for cultivating profitably the wide domain of Natural Philosophy.

By recasting the several treatises, by rejecting every proposition not essential, and by reducing and condensing the rest, I have succeeded in comprising, in this small volume, all that is truly valuable in Geometry. It is adapted for a course of three months, but is perfectly accessible, I trust, to the solitary Student, who shall peruse it with care and resolution. Geometrical Analysis, assimilating with Inductive Philosophy, forms the best preparation for exploring the sources of Physical Science.

The great error in modern education consists in the undue attention paid to the dead languages, which consumes the precious time that should be devoted, during the freshness

of youth, to the higher intellectual pursuits. The study of Mathematics becomes always more difficult and irksome, the longer it is deferred. The road to abstract science can certainly be smoothed; but we may rest assured, that any popular version is quite illusory, and that no portion of sound knowledge is ever acquired without some corresponding exertion of mind.

College of Edinburgh,
*5th May* 1828.

# RUDIMENTS

OF

# PLANE GEOMETRY.

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## INTRODUCTION.

NUMBER was framed to express merely the aggregation of individuals of the same class. But, from an extension of the principle of like composition, it came afterwards to represent Magnitude or Quantity in general. Quantities being viewed as the collection of certain elementary portions, *magnitude* was hence assimilated to *multitude*.

Those elementary portions or component units were often arbitrary, and commonly derived from the measurement of the human body. Thus, *the fathom*, *the yard*, *the cubit*, *the palm*, *the foot*, or *the inch*, were taken as standards of length; and from these originated the various superficial and solid measures. Water, being a fluid of uniform constitution, furnished by its bulk the basis of a system of weights.

In some cases, Nature suggested the simplest subdivision of Quantity. Thus, an *angle* is estimated from the partition of the circumference of a circle; and *time*, which bears some analogy to it, is reckoned by the periods of the revolution of the sun, the moon, or the stars.

If a Quantity cannot be formed completely, by repeating the usual standard, the application of a smaller measure may reach it more nearly; but the successive subdivisions of the unit will approach to any degree of accuracy.

It is possible, however, to *conceive* a Quantity so circumstanced, as to admit of no absolute measure, which by repetition shall exhaust the whole, without leaving any remainder. But this apparent imperfection, arising from the infinite variety attributed to possible magnitude, creates no real obstacle to the progress of accurate science. The measure, or primary element, being assumed successively smaller and smaller, its corresponding remainder must continually diminish. This progressive exhaustion will hence approximate nearer than any assignable difference to its final term.

It is obvious, that quantities of any kind, whether of length, extent, or weight, must have the same composition, when each contains its measure an equal number of times. But quantities viewed in pairs may be considered as having a similar composition, if the corresponding terms of each pair contain its mea-

sure equally. Two pairs of quantities of a similar composition, being thus formed by the same distinct aggregations of their elementary parts, constitute a *Proportion* or *Analogy*.

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To facilitate the investigation of the properties of *Numbers*, it is expedient to adopt a clear and concise mode of notation.

1. The sign $=$ expresses *equality*, $>$ *majority*, and $<$ *minority*. Thus, $A > B$ signifies that A is greater than B, and $A < B$ imports that A is less than B.

2. The signs $+$ and $-$, or *plus* and *minus*, mark the addition and subtraction of the quantities to which they are prefixed: Thus, $A + B$ denotes that B is to be joined to A, and $A-B$ signifies that B is to be taken away from A. Sometimes these two symbols are combined together: Thus, $A \pm B$ represents either the sum of A and B, or the excess of A above B.

3. To express multiplication, the quantities are placed close together; or they may be connected by the full point (.), or the St Andrew's cross $\times$ : Thus, AB, or A.B, or $A \times B$, denotes the product of A by B; and ABC indicates the result of the continued multiplication of A by B, and of this product again by C.

4. When the same number is repeatedly multiplied, the product is termed its *power;* and the number itself, in reference to that power, is called the *root.* The notation here may be still farther abridged, by retaining only a sin-

gle letter with a small figure placed over it, to mark how often it is understood to be repeated: This figure serves also to distinguish the order of the power. Thus AA, or $A^2$, signifies that A is multiplied by A, and that the product is the *second power* of A; and AAA, or $A^3$, in like manner, imports that AA is again multiplied by A, and that the result is the *third power* of A.

5. A *root* is denoted, by prefixing the contracted *r*, or the symbol $\surd$. Thus, $\surd A$ or $\sqrt[2]{A}$ marks the *second root* of A, or that number of which A is the second power; $\sqrt[3]{A}$ signifies the *third root* of A, or the number which has A for its third power.

6. To represent the multiplication of complex quantities, they are included by a parenthesis. Thus, $A(B+C-D)$ denotes that the amount of $B+C-D$, considered as a simple quantity, is multiplied into A.

7. Ratios and analogies are expressed, by inserting points in pairs between the terms. Thus $A : B$ denotes the ratio of A to B; and the compound symbols $A : B :: C : D$, signify that the ratio of A to B is the same as that of C to D, or that A is to B as C to D.

8. The result of the multiplication of two numbers is called their *product*, and, in reference to it, the numbers themselves are termed *factors*.

9. A *divisor* is the number contained in another called the *dividend*, the *quotient* expressing how often it is so contained.

The following Definitions are also required:

1. Quantities are *homogeneous*, which can be added or subtracted.

2. One quantity is said to *contain* another, when the repeated subtraction of the smaller leaves no remainder.

3. A quantity which is contained in another, is said to *measure* it.

4. The quantity which is measured by another, is called its *multiple;* and that which measures the other, its *submultiple.*

5. *Like* multiples and submultiples are those which contain their measures equally, or which measure equally their corresponding compounds.

6. Quantities are *commensurable*, which have a finite common measure; they are *incommensurable*, if they will admit of no such measure.

7. That relation which one quantity is *conceived* to bear to another regarding their composition, is named a *ratio.*

8. A *proportion* or *analogy* consists in the identity of ratios, or similarity in the composition of quantities.

9. A ratio is *direct*, if it follows the order of the terms compared; it is *inverse* or *reciprocal*, when it holds a reversed order.

Thus, if the ratio of A to B be considered as *direct*, the ratio of B to A is *inverse* or *reciprocal.*

10. Four quantities are said to be *proportional*, when a submultiple of the first is contained in the second as often as a like submultiple of the third is contained in the fourth.

11. Of proportional quantities, the first of each pair is named the *antecedent*, and the second the *consequent.*

12. Quantities form a *continued proportion*, when the intermediate terms stand in the double relation of consequents and antecedents.

13. The ratio which one quantity has to another, may be considered as *compounded* of all the connecting ratios among interposed quantities.

Thus, the ratio of A to D is viewed as *compounded* of that of A to B, that of B to C, and that of C to D.

14. A ratio combined *twice* forms a *duplicate ratio*, of which it again is termed the *subduplicate*.

15. When a proportion consists of three terms, the middle one is said to be a *mean proportional* between the two extremes.

16. Of quantities in a continued proportion, the first is said to have to the third, the *duplicate* ratio of what it has to the second; to have to the fourth, a *triplicate* ratio; to the fifth, a *quadruplicate* ratio; and so forth, according to the number of ratios introduced between the extreme terms.

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*Since all quantities can be expressed to any degree of accuracy by* Numbers, *it is obvious that the reasoning respecting* Numerical Proportions *must apply to* Proportions in General.

## PROPOSITION I.

Homogeneous quantities are proportional to their like multiples or submultiples.

Let A, B be two quantities of the same kind, and $pA$, $pB$ their like multiples; then $A : B :: pA : pB$.

For, since A and B are capable of being measured to any required degree of precision, suppose $a$ to be the measure which A and B contain respectively $m$ and $n$ times, or that $A = m.a$ and $B = n.a$. Whence $pA = p.ma = m.pa$, and $pB = p.na = n.pa$; and therefore $a$ and $pa$ are like submultiples of A and of $pA$, which contain them severally $m$ times; and these like submultiples are both contained equally, or $n$ times, in B and in $pB$. Wherefore the quantities A, B, and $pA$, $pB$ are proportional; and A, $pA$ are the antecedents, and B, $pB$, the consequents, of the analogy.

Again, because the ratio of $pA$ to $pB$ is thus the same as that of A to B, which, in reference to $pA$ and $pB$, are only like submultiples, it follows that homogeneous quantities are also proportional to their like submultiples.

## PROPOSITION II.

In proportional quantities, according as the first term is greater, equal, or less than the second, the third term is greater, equal, or less than the fourth.

Let $A : B :: C : D$; if $A > B$, then $C > D$; if $A = B$ then $C = D$; or if $A < B$, then $C < D$.

For, if A be greater than B, then the measure or sub-

multiple of A must be contained oftener in B, and hence the like submultiple of C will be contained oftener in D wherefore C is greater than D.

If A be equal to B, the measure of A is contained equally in B, and hence that of C in D, or C is equal to D.

But, if A be less than B, the measure of A is not contained so often in B, and hence the measure of C is not contained so often in D, or C is less than D.

## PROPOSITION III.

If four numbers be proportional, the product of the extremes is equal to that of the means; and, of two equal products, the factors are convertible into an analogy, of which these form severally the extreme and the mean terms.

Let A : B : : C : D; then AD = BC.

For (1.) A.D : B.D : : B.C : B.D; and the second term of this analogy being equal to the fourth, therefore (2.) AD = BC.

Again, let AD = BC; then A : B : : C : D.

For, by identity of ratios, AD : BD : : BC : BD, and hence (1.) A : B : : C : D.

*Cor.* Hence a proportion is not affected, by transposing or interchanging its extreme and mean terms.—It follows, therefore, 1. That the terms of an analogy may be *inverted*, or that the second is to the first, as the fourth to the third; and 2. That they may be *alternated*, or that the first is to the third, as the second to the fourth.

## PROPOSITION IV.

The terms of a numerical or homogeneous analogy are proportional by *composition ;* or the sum of the first and second is to the second, as the sum of the third and fourth to the fourth.

Let A : B : : C : D, then by *composition* A+B : B : : C+D : D.

Because A : B : : C : D, the product AD = BC (3.); add to each of these the product BD, and AD + BD = BC+BD. But AD + BD = D(A+B), and BC + BD = B(C + D); wherefore (3.) assuming the factors of these equal products for the extreme and mean terms, A+B : B : : C+D : D.

## PROPOSITION V.

The terms of a numerical or homogeneous analogy are proportional by *division;* or the difference of the first and second is to the second, as the difference of the third and fourth to the fourth.

Let A : B : : C : D; suppose A to be greater than B, then will C be greater than D (2.): It is to be proved that A — B : B : : C — D : D.

For, since A : B : : C : D, the product AD = BC (3.), and, taking BD from both, the compound product AD — BD is equal to BC — BD; wherefore, by

resolution, $(A - B)D = B(C - D)$, and consequently $A - B : B :: C - D : D$.

If B be greater than A, then $BD - AD = BD - BC$, and, by resolution, $(B - A)D = B(D - C)$; whence $B - A : B :: D - C : D$.

*Cor.* By inverting the two last propositions, it follows that the terms of an analogy are proportional by *conversion*; or the first is to the sum or difference of the first and second, as the third to the sum or difference of the third and fourth.

## PROPOSITION VI.

The terms of a numerical or homogeneous analogy are proportional, by *mixing;* or the sum of the first and second is to the difference, as the sum of the third and fourth to their difference.

Let $A : B :: C : D$, and suppose $A > B$; then $A + B : A - B :: C + D : C - D$.

For, by conversion, $A : A + B :: C : C + D$, and alternately $A : C :: A + B : C + D$.

Again, by conversion, $A : A - B :: C : C - D$, and alternately $A : C :: A - B : C - D$. Whence, by identity of ratios, $A + B : C + D :: A - B : C - D$, and alternately $A + B : A - B :: C + D : C - D$.

The same reasoning will hold if A be less than B, the order of these terms being only changed.

## PROPOSITION VII.

If the means or extremes be the same, or alternate in two analogies, a third one may be obtained, by interchanging those terms.

1. Let $A : B :: C : D$, and $E : B :: C : F$; then $A : E :: F : D$.

For, since $A : B :: C : D$, $AD = BC$; and because $E : B :: C : F$, $EF = BC$. Whence $AD = EF$, and $A : E :: F : D$.

2. Let $A : B :: C : D$, and $E : A :: D : F$; then $B : E :: F : C$.

For, from the first analogy, $AD = BC$, and, from the second, $EF = AD$; whence $BC = EF$, and consequently $B : E :: F : C$.

## PROPOSITION VIII.

If there be any number of homogeneous proportionals, as one antecedent to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let $A : B :: C : D :: E : F :: G : H$; then $A : B :: A+C+E+G : B+D+F+H$.

Because $A : B :: C : D$, (3.) $AD = BC$; and, since $A : B :: E : F$, $AF = BE$, and, for the same reason, $AH = BG$. Consequently, the aggregate products, $AB+AD+AF+AH = BA+BC+BE+BG$; and, by resolution, $A(B + D + F + H) = B(A + C + E+G)$; whence $A : B :: A+C+E+G : B+D+F+H$.

## PROPOSITION IX.

If two analogies have the same terms as antecedents or as consequents, a third analogy may be formed, by interchanging those pairs of terms.

Let $A : B :: C : D$ and $A : E :: C : F$; then $B : E :: D : F$; or if $A : B :: C : D$, and $B : E :: D : F$; then $A : E :: C : F$.

For, by alternating those analogies, the identity of the ratios will appear, which, by applying alternation, will produce the new analogy.

## PROPOSITION X.

In continued proportionals, the difference between the first and second is to the first as the difference between the first and last terms to the sum of all the terms excepting the last.

Let $A : B :: B : C :: C : D :: D : E$; then, if $A > B$ $A - B : A :: A - E : A+B+C+D$.

For (1.) $A : B :: A + B + C + D : B + C + D + E$, and consequently $A - B : A :: (A + B + C + D) - (B + C + D + E) : A+B+C+D$; that is, omitting $B+C+D$ in the third term, $A-B : A :: A-E : A+B+C+D$.

If $A \angle B$, then $B - A : A :: (B + C + D + E) - (A+B+C+D) : A+B+C+D$, that is, $B-A : A :: E - A : A+B+C+D$.

## PROPOSITION XI.

The products of the like terms of any numerical proportions, are themselves proportional.

Let A : B : : C : D
E : F : : G : H
I : K : : L : M;

then AEI : BFK : : CGL : DHM.

For (2.), from the first analogy AD = BC, from the second analogy EH = FG, and from the third analogy IM = KL; whence the compound product AD.EH.IM = BC.FG.KL. But AD.EH.IM = AEI.DHM, and BC.FG.KL = BFK.CGL; wherefore AEI.DHM = BFK.CGL, and consequently (2.) AEI : BFK : : CGL : DHM.

## PROPOSITION XII.

The ratio which is conceived to be compounded of other ratios, is the same as that of the products of their corresponding numerical expressions.

Suppose the ratio of A : D is compounded of A : B, of B : C, and of C : D, and let A : B : : K : L, let B : C : : M : N, and let C : D : : O : P; then will A : D : : KMO : LNP.

For, since A : B : : K : L,
B : C : : M : N,
and C : D : : O : P,

the products of the similar terms are proportional; or ABC : BCD : : KMO : LNP. But A : D : : ABC : BCB (3.), and consequently A : D : : KMO : LNP.

## PROPOSITION XIII.

A duplicate ratio is the same as the ratio of the second powers of the terms of its numerical expression, and a triplicate ratio is the same as that of the third powers of those terms.

Let $A : B :: B : C :: C : D$; then $A^2 : B^2 :: A : C$, and $A^3 : B^3 :: A : D$.

For, since $A : B :: B : C$,
and $A : B :: A : B$, the products of the corresponding terms are proportional, (11.) or $A^2 : B^2 :: BA : CB$. Whence (1.) $A^2 : B^2 :: A : C$.

Again, since $A : B :: B : C$,
and $A : B :: C : D$,
and $A : B :: A : B$,

$A^3 : B^3 :: BCA : CDB$. And consequently, by combination, $A^3 : B^3 :: A : D$.

## PROPOSITION XIV.

Given two homogeneous quantities, to find, if possible, their greatest common measure.

Let it be required to find the greatest common measure, which two quantities A and B, of the same kind, will admit. Assuming A to be greater than B, take B repeatedly out of A, till the remainder C be less than it; again, take C by succession out of B, till there remain only D; and continue this decomposition, till the last divisor, suppose E, leave no remainder whatever; E is the greatest common measure of the quantities proposed.

For, because the quantity sought measures B, it will measure its multiple; and since it also measures A, it must measure evidently the difference between the multiple of B and A, that is, C; the required measure, therefore, measures the multiple of C, and consequently the difference of this multiple from B, that is, D: And lastly, this measure, since it measures the multiple of D, must measure the difference of this from C, or E. Conceiving the decomposition to terminate here, the common measure of A and B, since it measures E, must be E itself; and it is also the greatest possible measure, for nothing greater than E can contain it.

By retracing the steps likewise, it might be shown, that E actually measures the whole train, the preceding terms D, C, B, and A.

If the process of decomposition should never terminate, the quantities A and B do not admit of a common measure,—or they are *incommensurable*. But, as the residue of the subdivision is necessarily diminished at each step of this operation, it is evident that some element may always be discovered, which will measure A and B nearer than any assignable limit.

## PROPOSITION XV.

To express by numbers, either exactly or approximately, the ratio of two given homogeneous quantities.

Let A and B be two quantities of the same kind, whose numerical ratio it is required to discover.

Find, by the last proposition, the greatest common measure E of the two quantities; and let A contain this mea-

sure K times, and B contain it L times: Then will the ratio K : L express the ratio of A : B.

For the numbers K and L severally consist of as many units, as the quantities A and B contain their measure E. It is also manifest, since E is the greatest possible divisor, that K and L are the smallest numbers capable of expressing the ratio of A to B.

If A and B be incommensurable quantities, their decomposition is capable at least of being pushed to an unlimited extent; and, consequently, a divisor can always be found so extremely minute, as to measure them both to any degree of precision.

## PROPOSITION XVI.

The portions of a number are incommensurable, of which the second power of the one is equal to the products of the whole and the remainder.

Let A be any number divided into two parts B and C, such that $B^2 = AC$; these are incommensurable portions.

For put A—B = C, and B—C = D. Since $B^2=AC$, take BC or CB from both, and $B^2 - BC = AC - CD$, or $B(B-C) = C(A-D)$; that is, $BD = C^2$. Wherefore, when B is taken out of A, the same relation subsists between C and D, the portions of the remainder B. And it follows that, repeating the operation, the second power of the greater portion would, at each step, be equal to the product of each successive whole and its remainder. This process of decomposition would hence never terminate; and therefore no final divisor, however small, can exist, which would render the numbers commensurable.

# RUDIMENTS

OF

# PLANE GEOMETRY.

## BOOK I.

GEOMETRY is that branch of natural science which treats of bounded space.

Our knowledge of external objects is received through the medium of the senses. The science of Physics considers Bodies as they actually exist, invested with all their qualities: Its researches are hence guided by that refined species of observation which is termed Experiment. But Geometry takes a more confined view; and, selecting the single property of *Magnitude*, it can safely pursue the most lengthened train of investigation. It contemplates merely the external forms of bodies, and the spaces

which they occupy. Geometry is thus founded likewise on observation; but of a kind so familiar and obvious, that the primary notions which it furnishes might seem intuitive. This science, proceeding from a basis of extreme simplicity, is therefore supereminently distinguished, by the luminous evidence which attends every step of its progress.

## PRINCIPLES.

In contemplating an external object, we can, by repeated acts of abstraction, reduce the complex notion that arises in the mind into others which are successively simpler. *Body*, divested of all its essential characters, presents the idea of mere *surface*; a *surface*, considered distinct from its peculiar qualities, exhibits only *linear boundaries*; and a *line*, if we throw aside its continuity, leaves nothing in the imagination but the *points* which denote its extremities. A solid is bounded by *surfaces*; a surface is circumscribed by *lines*; and a line is terminated by *points*. A point assigns *position*; a line marks *distance*; and a surface shews *extension*. A line has *length* only; a surface has both *length* and *breadth*; and a solid combines all the three dimensions of *length*, *breadth* and *thickness*.

The uniform tracing of a line, which is stretched through its whole extent in the same direction, gives the idea of a *straight* line. No more than one straight line can hence join two points; and if a straight line be conceived to turn like an axis about both extremities, its intermediate points will not change their position.

From our idea of a straight line is derived that of a *plane* surface, which has a like uniformity of character. A straight line connecting any two points situate in a plane, lies wholly on the surface; and consequently planes must admit a mutual application.

*Two* points indicate the position of a straight line; for the line may be conceived to turn about one of those points till it falls upon the other. But to assign the position of a plane, it requires *three* points; because a plane touching the straight line which joins two of the points, may be made to revolve, till it meets the third point.

The separation or opening of two straight lines at their point of intersection, constitutes an *angle*. If we obtain the idea of *distance*, or linear extent, from the inspection of *progressive* motion, we derive that of *divergence*, or angular magnitude, from the consideration of *revolving* motion.

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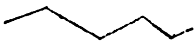
Geometry is divided into Plane and Solid; the former confining its views to the properties of space figured on the same plane; the latter embracing the relations of different planes or surfaces, and of the solids which these may describe or terminate. In the following definitions, therefore, the points and lines are all considered as existing in the same plane.

# DEFINITIONS.

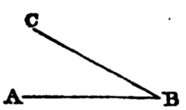
1. A *crooked* line is that which consists of straight lines not continued in the same direction.



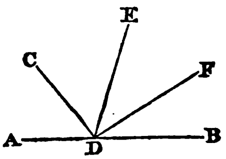
2. A *curved* line is that of which no portion is a straight line.

3. The straight lines which contain an *angle* are called its *sides*, and their point of origin, or intersection, its *vertex*.

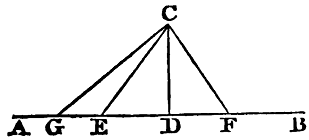
To abridge the reference, it is usual to intimate an angle by tracing its sides; the letter at the vertex, which is common to them both, being put in the middle. Thus the angle contained by the straight lines or sides AB and BC, or the opening formed by turning BA about the point or vertex B into the direction BC, is designed ABC or CBA.



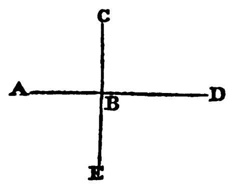

The angle ADC may successively widen or enlarge into ADE, ADF, till, making half a revolution, it expands into ADB, being then contained by DA and DB, the opposite portions of the same straight line.



A straight line turning about the point C, and cutting AB, will successively augment the exterior angle, in travelling from A towards B. For it is obvious, that while the side GB slides along AB, the other side GC must open or enlarge the angle CGB, to allow it, from the advanced point E, to reach to C, with the angle CEB. This extreme angle successively expands into CDB and CFB.



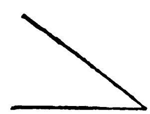

When a straight line, revolving about the point B, comes into the position CBE, such that the angles CBA, CBD, and consequently ABE and DBE, are all equal, it evidently quarters the whole circuit.




4. The fourth part of the entire revolution made by a straight line is termed a *right angle*.

5. The sides of a right angle are said to be *perpendicular* to each other.

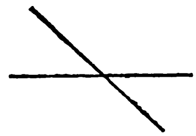
6. An *acute* angle is less than a right angle.



7. An *obtuse* angle is greater than a right angle.

8. One side of an angle forms with the other side produced, a *supplemental* or *exterior angle*.

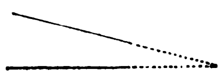
9. A *vertical* angle is formed by the production of both its sides.



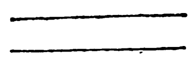
Vertical angles, having the same opening, are consequently equal.

10. A *reverse* angle is produced by the inverted opening of its sides, being composed of the vertical and two exterior angles.

11. Two straight lines are said to be *inclined* to each other, if they meet when produced; and the angle so formed is called their *inclination*.



12. Straight lines which have no mutual inclination, are termed *parallel*.

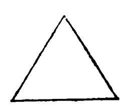


13. A *figure* is a plane surface included by a linear boundary called its *perimeter*.

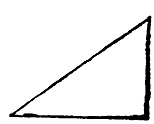
14. Of rectilineal figures, the *triangle* is contained by *three* straight lines.

15. An *isosceles* triangle is that which has two of its sides equal.

16. An *equilateral* triangle is that which has all its sides equal.

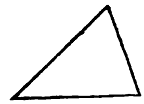


17. A *right-angled* triangle is that which has a right angle.



18. An *obtuse* angled triangle is that which has an obtuse angle.

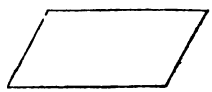
19. An *acute* angled triangle is that which has *all* its angles acute.



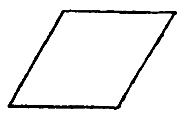
20. Any side of a triangle may be called its *base*, and the opposite angular point its *vertex*.

21. A *quadrilateral* figure is contained by *four* straight lines.

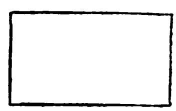
22. Of quadrilateral figures, a *rhomboid* has its opposite sides equal:



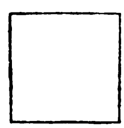
23. A *rhombus* has all its sides equal:



24. An *oblong*, or *rectangle*, has a right angle, and its opposite sides equal:

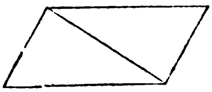


25. A *square* has a right angle, and all its sides equal:



26. A quadrilateral figure, of which the opposite sides are parallel, is called a *parallelogram*.

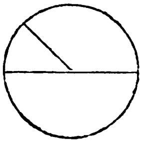
27. A straight line joining the opposite angular points of a quadrilateral figure, is named a *diagonal.*



28. A rectilineal figure having more than four sides, bears the general name of a *polygon.*

29. A *circle* is a figure described by the revolution of a straight line about one of its extremities:

30. The fixed point is called the *centre* of the circle, the describing line its *radius,* and the boundary traced by the remote end of that line its *circumference.*



31. The *diameter* of a circle is a straight line drawn through the centre, and terminated both ways by the circumference.

It is obvious that all radii of the same circle are equal to each other and to a semidiameter. It likewise appears, from the slightest inspection, that a circle can have only one centre, and that circles are equal which have equal diameters.

32. Figures are said to be *equal,* when, being applied to each other, they wholly coincide; they are *equivalent,* if, without coinciding, they yet contain the same space.

A PROPOSITION is a distinct portion of abstract science: It is either a *problem* or a *theorem*.

A PROBLEM proposes to effect some combination.

A THEOREM advances some truth, which is to be established.

A *problem* demands *solution*, a *theorem* requires *demonstration*; the former implies some operation, and the latter commonly proceeds by the aid of a construction.

A *direct* demonstration descends from the premises by a regular deduction.

An *indirect* demonstration attains its object, by showing that any other supposition than the one advanced would involve a contradiction.

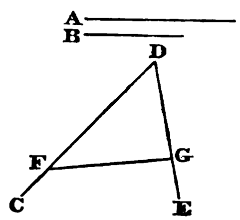
A COROLLARY is an obvious consequence resulting from a proposition.

*The operations in Geometry suppose the drawing of straight lines and the description of circles, or they require in practice the use of the rule and compasses.*

## PROPOSITION I. PROBLEM.

Two sides of a triangle, with their contained angle being given, to construct the triangle.

Let the straight lines A and B mark the lengths of two sides, and CDE represent their contained angle.




Take DF and DG respectively equal to A and B; and the straight line FG, which joins those extreme points, must evidently form the base of the triangle required.

*Cor.* Two sides and the angle contained by them will hence determine the identity of triangles, and triangles must be equal which have those mutual equalities.

*An angle may, in practice, be transferred by means of a jointed ruler, which opens at pleasure.*

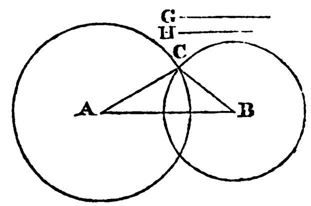
## PROP. II. PROB.

To construct a triangle, of which the three sides are given.

Let AB represent the base, and G, H two sides of the triangle, which it is required to construct.

From the centre A, with the distance G, describe a circle; and, from the centre B, with the distance H, describe

another circle, cutting the former in the point C; join AC and BC, and ACB is the triangle required.




Because all the radii of the same circle are equal, AC is equal to G; and, for the same reason, BC is equal to H. Consequently the triangle ACB answers the conditions of the problem.

If the radii G and H be equal to each other, the triangle will evidently be isosceles; and if those lines be likewise equal to the base AB, the triangle must farther be equilateral.

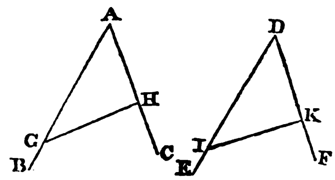
*Cor.* Hence triangles are equal, which have their sides respectively equal.

## PROP. III. PROB.

A side being given, to form an angle equal to a given angle.

From D the end of DE, to annex another side DF, making an angle equal to BAC.

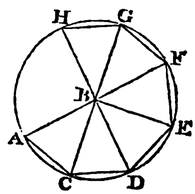
In AB and AC assume the points G and H, join GH, from DE cut off DI equal to AG, and upon DI construct (I. 2.) a triangle DIK, having its sides DK and IK equal respectively to AH and GH; IDK or EDF is the angle required.




For the triangle DKI (I. 2. cor.) admitting a perfect adaptation with AHG, the angle IDK must be equal to GAH.

This construction will become simpler, if the segments AG and AH be taken equal.

*Cor.* By the repeated application of this problem, an angle may be continually multiplied. For, having described a circle about the vertex B, and inserted the base AC successively in the circumference, a series of equal angles CBD, DBE, EBF, EBG, EBH, &c. will be constituted; so that the compound angles ABD, ABE, ABF, ABG, ABH, &c. are severally the double, the triple, the quadruple, the quintuple, the sextuple, &c. of the angle ABC.




It is farther evident, that such several additions have no limit, and that the angle so formed may continue to spread out its opening side, and make even repeated revolutions.

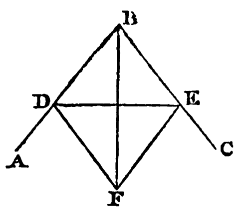
*In field operations, an angle is transferred by a Triangle of Cord, tied at the vertex, and knotted at the ends of the base; and hence, by the application of Proposition I., an inaccessible distance can be measured.*

## PROP. IV. PROB.

### To bisect a given angle.

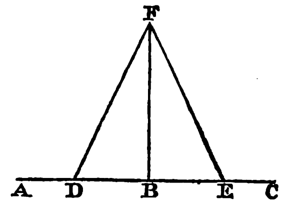
Let it be required to divide the angle ABC into two equal angles.

In the side AB take any point D, from BC cut off BE equal to BD; join DE, on which, assuming equal sides DE and EF, construct (I. 2.) the opposite isosceles triangle DFE; the straight line BF, which joins their vertices, bisects the angle ABC.




For the two triangles DBF and EBF, having the side DB equal to EB, the side DF to EF, and BF common to both, would admit a perfect adaptation, and consequently the angle DBF is equal to EBF.

*Cor.* Hence the mode of drawing a perpendicular from a given point B in the straight line AC; for the angle ABC, which the opposite segments BA and BC make with each other, being equal to two right angles, a straight line that bisects it must be the perpendicular required. Take BD equal to BE, and on the base DE construct the isosceles triangle DFE; the straight line BF, which joins the vertex of the triangle, is perpendicular to AC.



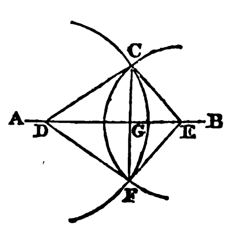

*Hence a perpendicular may be traced on the ground by a cord DFE, tied at F, and knotted at the equal distances D and E, and at the middle B.*

## PROPOSITION V.

To let fall a perpendicular upon a straight line, from a given point above it.

From the point C, to let fall a perpendicular upon the given straight line AB.

In AB take, any where towards A, the point D, and with the distance DC describe a circle; and, in the same line, take towards B another point E, and with the distance EC describe a second circle intersecting the former in the point F; join CF, crossing the given line at G: CG is perpendicular to AB.




For the straight lines DC, DF and EC, EF being joined, the triangles DCE and DFE have the side DC equal to DF, EC to EF, and DE common to them both; whence (I. 2. cor.) the angle CDE or CDG is equal to FDE or FDG. And because,

in the triangles DCG and DFG, the side DC is equal to DF, DG common, and the contained angles CDG and FDG are proved to be equal; these subordinate triangles are (I. 1.) equal, and consequently the angle DGC is equal to DGF, and each of them a right angle, or the line CG is perpendicular to AB.

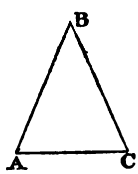
The construction will be rendered simpler in practice, by taking CE equal to CD, or describing a circle from C through the points D and E, the two centres of the intersecting circles.

## PROP. VI. THEOR.

**A triangle is isosceles, which has equal angles at its base; and conversely, the angles at the base of an isosceles triangle are equal.**

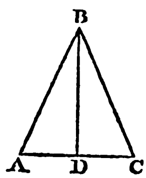
Let the triangle ABC have two equal angles BCA and BAC, the opposite sides BA and BC are likewise equal.

For conceive a copy of this triangle to be inverted and laid over it; the point C being applied to A, the point A will evidently fall upon C; and while the side CB now stretches along AB, the side AB will stretch along CB. Those sides must therefore meet in the same vertex B; whence the side BC, from its entire coincidence, must be equal to BA.




Again, the isosceles triangle ABC has the angles, BAC and BCA, at its base equal.

For (I. 4.) draw BD, bisecting the vertical angle ABC. Because the angle CBD is equal to ABD, the side BD common, and BC equal to BA, the triangle DBC, when turned over, would (I. 1. cor.) adapt itself to DBA; and consequently the angle BAD or BAC is equal to BCD or BCA.



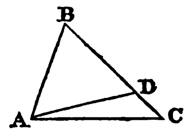

*Corollary.* Hence every equilateral triangle is equiangular, and conversely.

## PROP. VII. THEOR.

**In any triangle, that angle is the greater which is subtended by a greater side; and conversely, that side is the greater which lies opposite to a greater angle.**

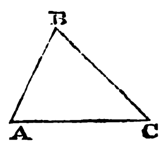
If the side BC, of the triangle ABC, be greater than BA, the angle BAC, which it subtends, is greater than BCA.

For let the part BD be taken equal to BA, and join AD. The triangle ABD being isosceles, the angle BAD is (I. 6.) equal to BDA; and the angle BAC, being evidently greater than the former, must likewise be greater than BDA. But this angle again is greater than the interior angle BCA, (corollary to definition 3.); wherefore the angle BAC is still greater than BCA.




Again, if the triangle have its angle CAB greater than ACB, the opposite side BC is also greater than AB.

For, if BC be not greater than AB, it must either be equal to AB, or less than it. But these sides cannot be equal, for the angles CAB and ACB would then be likewise equal; nor can BC be less than AB, for AB would be greater than BC, and the angle ACB greater than CAB, or CAB less than ACB, which is absurd. The side BC is thus neither equal to AB, nor less than it, and is therefore greater than AB.

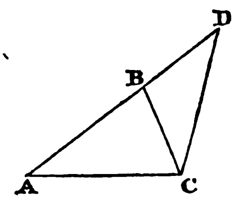



*Cor.* Hence the nearest distance from a point to a straight line is a perpendicular, and the radiants approaching this are the shorter.

## PROP. VIII. THEOR.

Two sides of a triangle are together greater than the base.

The sides AB and BC, of the triangle ABC, are together greater than AC the base, or third side.




For produce AB to D, till the addition BD be equal to BC, and join CD.

The triangle CBD being thus isosceles, the angle DCB (I. 6.) is equal to CDB; but the angle DCA is evidently greater than DCB, and consequently greater than CDB or CDA. Wherefore, in the triangle ADC, the side AD (I. 2.) is greater than AC; that is, AB and BD, or AB and BC conjoined, are greater than the base AC.

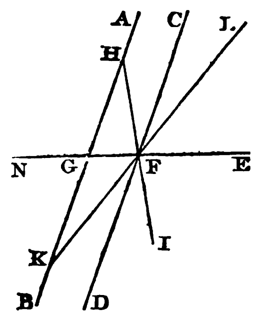
*Cor.* By applying this proposition, it is shown generally that the nearest distance between two points is a straight line.

## PROP. IX. THEOR.

If a straight line cut two parallels, it will form with them an exterior, equal to the interior, angle; and conversely, if an incident line make those angles equal, their two corresponding sides will be parallel.

If EFG cut the parallels CD and AB, the exterior angle EFC will be equal to the interior EGA; or if the angles EGA and EFC be equal, the lines AB and CD are parallel.

For imagine a straight line, extended indefinitely both ways, to turn gradually about the centre F, beginning from the position EFN, and advancing towards A, it will evidently meet AG the upper part of AB somewhere in H; but this point of concourse will constantly retire along GA, till it vanish beyond A; the other branch of the revolving line will now meet GB the under part of




AB at K, till, carrying its section always higher, it comes into the first position EFN. So long as this incident line cuts AB *above* G, it makes the exterior angle EFH (Def. 8. cor.) *greater* than EGA; but when it comes to cut AB *below* G, the exterior angle EFL, or its vertical KFN, is (for the same reason) *less* than KGN or the vertical EGA. But, in passing through all the intermediate gradations, the line EGN must attain a certain position CFD, where the exterior angle EFC is just equal to EGA, and where EGN does not meet the extension of AB either above or below the point G. The exterior angle EFC, of the parallel CD, is thus equal to the interior EGA. But CFD is the only line drawn through F that can produce such equality, which is hence the criterion of parallelism; for the slightest deviation from that individual direction would occasion a concurrence with AB.

*Cor.* Since the exterior angle EFC is equal to its vertical DFG, this angle must be equal to the interior or alternate angle FGA. Again, to the equal angles AGF and CFE add CFG, and the two interior angles AGF

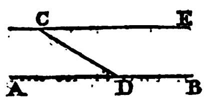
and CFG are together equal to CFE and CFG, which make two right angles. These equalities must likewise determine the parallelism of two lines.

## PROP. X. THEOR.

Through a given point, to draw a straight line parallel to a given straight line.

To draw, through the point C, a straight line parallel to AB.

In AB take any point D, join CD, and at the point C make (I. 4.) an angle DCE equal to CDA; the line CE is parallel to AB.



For the angles CDA and DCE, thus formed equal, are the alternate angles which CD makes with the straight lines CE and AB, and therefore, by the last proposition, these lines are parallel.

The corollary to the last proposition suggests likewise two other simple constructions.

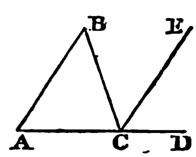
## PROP. XI. THEOR.

An exterior angle of a triangle is equal to both its opposite interior angles, and all the interior angles are together equal to two right angles.

The exterior angle BCD, formed by the production of the side AC of the triangle ABC, is equal to the two opposite interior angles CAB and CBA, and all the interior

angles CAB, CBA, and BCA of the triangle are together equal to two right angles.

For having through the point C drawn the straight line CE parallel to AB; the interior angle BAC is (I. 9.) equal to the exterior one ECD, and, for the same reason, the alternate angle ABC is equal to BCE. Wherefore the two angles CAB and ABC are equal to DCE and ECB, which together form the whole exterior angle BCD.




Now, to the two interior angles CAB and ABC, and to the exterior angle BCD, add the adjacent angle BCA; and all the interior angles of the triangle ABC are equal to BCD and BCA, or two right angles.

*Cor.* Hence the two acute angles of a right-angled triangle are equal to one right angle; and hence each angle of an equilateral triangle is two-third parts of a right angle.

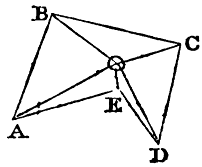
## PROP. XII. THEOR.

The interior angles of a rectilineal figure are equal to twice as many right angles, abating four, as the figure has sides.

The angles ABC, BCD, CDE, DEA, (a reverse angle in this figure,) and EAB are collectively equal to four right angles.

For assume a point O within the figure, and draw straight lines OA, OB, OC, OD, and OE to the several

corners. It is obvious, that the figure is thus resolved into as many triangles as it has sides, and the aggregate angles must, by the last proposition, be equal to twice as many right angles. But the angles ABO, OBC; BCO, OCD; CDO, ODE; DEO, OEA; and EAO, OAB at the bases of these triangles constitute the internal angles of the figure. Consequently, from the whole amount, are to be deducted the vertical angles about the point O, which are equal to four right angles.



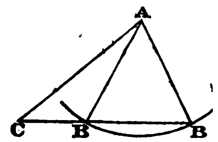

*Cor.* Hence the exterior angles of a rectilineal figure are equal to four right angles.

## PROP. XIII. THEOR.

Two sides and an opposite angle being given, to construct the triangle.

The sides AB, AC, and the angle ACB opposite to the former, being given, to construct the triangle.

From A as a centre, with a distance equal to the shorter side AB, describe a circle meeting BC, or cutting it in the two points B, B′, join AB or AB′, and the triangle ABC or AB′C will evidently combine the required conditions.

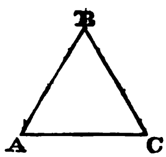



Unless, therefore, the other opposite angle ABC be a right angle, the triangle will have two phases; and according as that angle is acute or obtuse, its vertex must occupy the position B or B′.

## PROP. XIV. THEOR.

**The angles of a triangle and a side being given, to construct the triangle.**

Let the side AC with two angles, and consequently the third one be given, to construct the triangle.




From the extremities A and C of the given side, draw AB and BC containing likewise the given angles. These straight lines will evidently meet in the vertex B, and form the required triangle.

*General Corollary.* These four problems then,—the 1, 2, 15 and 16 comprise all the conditions requisite for determining a triangle, and they are reducible to a very simple enunciation. The sides of a triangle are obviously independent of each other,—subject merely to this condition, that any one of them be less than the remaining two sides. But all the angles of a triangle being equal to two right angles, the third angle is evidently derived from the other two. A triangle has, therefore, only five original and variable parts—the three sides and two of its angles. *Any three of these parts being ascertained, the triangle is absolutely determined.* Thus—when (1.) all the three sides are given, —when (2.) two sides and their contained angle are given, —when (4.) two sides and an opposite angle are given, with the affection of the triangle, or when (3.) one side and two angles, and thence the third angle are given,— the triangle is completely marked out.

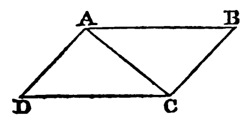
Triangles are therefore equal which have those three parts respectively equal.

## PROP. XV. THEOR.

The opposite sides of a rhomboid are parallel.

If the opposite sides AB, DC, and AD, BC of the quadrilateral figure ABCD be equal, they are also parallel.

For draw the diagonal AC. And because the side AB is equal to DC, BC to AD, and AC is common to the two triangles ABC and ADC; these triangles are (I. 2.) equal. Whence the angle ACD is equal to CAB, and therefore the side AB of the rhomboid is (I. 9.) parallel to CD; and, for the same reason, the angle CAD being equal to ACB, the side AD is parallel to BC.



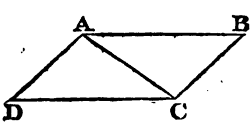

*Hence the composition of the common parallel ruler.*

## PROP. XVI. THEOR.

The opposite sides and angles of a parallelogram are equal.

Let the quadrilateral figure ABCD have the sides AB and BC parallel to CD and AD; these are respectively equal, and so are likewise the opposite angles at A and C, and at B and D.

For draw the diagonal AC. Because AB is parallel to CD, the alternate angles BAC and ACD are (I. 9.) equal; and since AD is parallel to BC, the alternate angles ACB and CAD are likewise equal. Wherefore the triangles ABC and ADC, having



the angles CAB and ACB equal to ACD and CAD, and the interjacent side AC common to both, are (I. 14. cor.) equal. Consequently, the side AB is equal to CD, and the side BC to AD; and these opposite sides of the rhomboid being thus equal, the opposite angles (I. 12.) must be likewise equal.

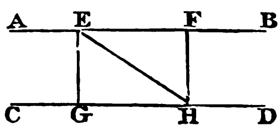
*Cor.* Hence the diagonal divides a rhomboid or parallelogram into two equal triangles.

## PROP. XVII. THEOR.

Parallels are equidistant, and equidistant lines parallel.

The perpendiculars EG, FH, let fall from any points E, F in the straight line AB, upon its parallel CD, are equal; and if these perpendiculars be equal, the straight lines AB and CD are parallel.

For join EH. The right angle EGH being equal to FHG, or its adjacent angle FHD, the perpendiculars EG and FH are (I. 9.) parallel, and consequently the alternate angles HEG and EHF are equal.




But EF being parallel to GH, the alternate angles EHG and HEF are likewise equal; and thus the two triangles HGE and HFE, having the angles HEG and EHG respectively equal to EHF and HEF, and the side EH common to both, are (I. 14. cor.) equal, and hence the side EG is equal to FH.

Again, if the perpendiculars EG and FH be equal, the two triangles EGH and EFH, having the side EG equal to FH, EH common, and the contained angle HEG equal to EHF, are (I. 1. cor.) equal, and therefore the angle

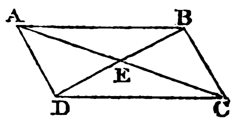
EHG equal to HEF, and (I. 9. cor.) the straight line AB parallel to CD.

*Cor.* Hence lines parallel to the same straight line are likewise parallel to each other.

## PROP. XVIII. THEOR.

The diagonals of a parallelogram are mutually bisected.

The diagonals AC and BD of the parallelogram ABCD, crossing each other, make AE equal to EC, and BE equal to ED.

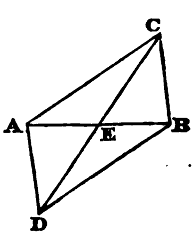



For the triangles AEB and CED having equal bases AB and CD, and the angles EAB and EBA equal to the alternate angles ECD and EDC, are equal (I. 16. cor.). Whence the sides AE and BE are equal to EC and ED.

## PROP. XIX. PROB.

To bisect a given straight line.

Let it be required to bisect AB.




Assume any lateral point C, join AC and BC, and upon the base AB construct the opposite triangle ADB, having its sides BD and AD respectively equal to these lines, and connect CD, which will cut AB in the middle point E.

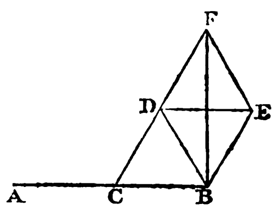
For the figure ACBD is obviously a rhomboid or parallelogram, and its diagonal CD must hence (I. 18.) bisect AB.

The construction will be simplified, if AC be taken equal to BC.

## PROP XX. PROB.

To draw a perpendicular from the extremity of a given straight line.

In AB take any point C, and on BC describe an equilateral triangle CDB, on its side DB, another DEB; and on DE the side of this, a third equilateral triangle DFE; join the last vertex F with the point B: and BF is the perpendicular required.




Because the triangles CDB and DBE are equilateral, the angles CBD and DBE are each of them equal to two-third parts of a right angle, (I. 13. cor.) and the triangles BDF, BEF, having the sides BD, DF equal to BE, EF, and the side BF common, are equal, and consequently the angles FBD and FBE are equal, and each of them the half of DBE. The angle FBD being therefore one-third part of a right angle, and the angle DBA two-third parts, the whole angle FBC must be an entire right angle, or the straight line BF is perpendicular to AB.

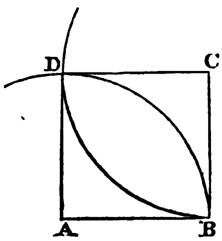
*This construction is often more convenient in practice than the one given as a corollary to Proposition IV.*

## PROP. XXI. PROB.

On a given straight line, to construct a square.

Let AB be the side of the square which it is required to construct.

From the extremity B draw, by the last proposition, BC perpendicular to BA and equal to it, and, from the points A and C with the distance AB or CB describe two circles intersecting each other in the point D, join AD and CD; the quadrilateral figure ABCD is the square required.




For, by this construction, the figure has all its sides equal, and one of its angles ABC a right angle, which comprehends the whole definition of a square.

## RUDIMENTS

OF

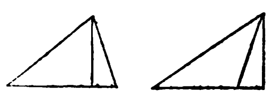
# PLANE GEOMETRY.

## BOOK II.

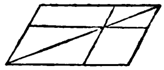
### DEFINITIONS.

1. In a right-angled triangle, the side that subtends the right angle is termed the *hypotenuse;* either of the sides which contain it, the *base;* and the other side the *perpendicular.*

2. The *altitude* of a triangle is a perpendicular let fall from the vertex upon the base or its extension.



3. The *complements* of rhomboids about the diagonal of a rhomboid, are the spaces required to complete the rhomboid; and the defect of each rhomboid from the whole figure, is termed a *gnomon.*



4. A rhomboid or rectangle is said to be *contained* by any two adjacent sides.

A rhomboid is often indicated merely by the two letters placed at opposite corners.

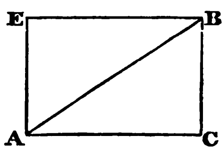
## PROP. I. THEOR.

**A triangle is equivalent to half the rectangle under its base and altitude.**

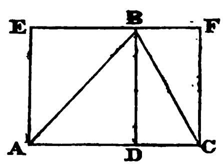
The triangle ABC is half of the circumscribing rectangle AEFC.

For the perpendicular from the vertex of the triangle, may meet the extremity of the base AC, or fall within or without the base.

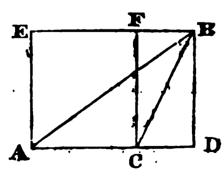
In the first case, the diagonal AB (I. 17. cor.) divides the rectangle AEBC, contained by the base AB and altitude, BC into two equal portions, of which the triangle ABC is the lower.




When the perpendicular falls within the base AC, the triangles ABD and CBD are (I. 17. cor.) halves of the rectangles AEBD and CFBD; and, therefore, the whole triangle ABC is the half of the combined rectangle AEFC.




Lastly, when the perpendicular BD falls beyond the base, the triangle ABD is half of the rectangle AEBD, and the triangle CBD is half of the rectangle CFBD. Wherefore, the excess of the former, or the triangle ABC, is half of the remaining rectangle AEFC.



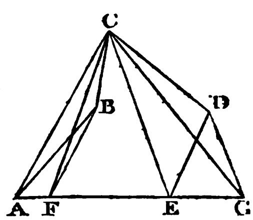

*Cor.* Parallelograms on the same or equal bases, and between the same parallels, are equivalent; for their diagonals bisect them into triangles having the same base and altitude.

## PROP. II. PROB.

To form a triangle equivalent to any rectilineal figure.

Let it be required to reduce the five-sided figure ABCDE to a triangle, or to find a triangle that shall contain an equal space.

Join any two alternate points A, C, and through the intermediate point B, draw BF parallel to AC, meeting either of the adjoining sides AE or CD in F, and join CF. Again, join the alternate points C, E, and through the intermediate point D draw the parallel DG, to meet either of the adjoining sides AE or BC, or their production in G, and join CG. The triangle FCG is equivalent to the five-sided figure ABCDE.




Because the triangles CFA and CBA have by construction the same altitude and stand on the same base AC, they are (II. 1.) equivalent; take each of them away from the space ACDE, and there remains the quadrilateral figure FCDE equivalent to the five-sided figure ABCDE. Again, because the triangles CDE and CGE are equivalent, having the same altitude and the same base; add the triangle FCE to each, and the triangle FCG is equivalent to the quadrilateral figure FCDE, and consequently to the original figure ABCDE.

In this manner, any polygon may, by successive steps, be reduced to a triangle; for an exterior triangle such as

CDE, or an interior one such as ABC, is always exchanged for another equivalent, which, attaching itself to either of the adjoining sides, coalesces with the rest of the figure.

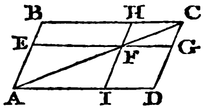
*This problem is of singular use in practice, since it enables the surveyor to abridge his computations by reducing at once a plan delineated into an equivalent triangle.*

## PROP. III. THEOR.

The complements of the rhomboids about the diagonal of a rhomboid, are equivalent.

Let EI and HG be rhomboids about the diagonal of the rhomboid BD; their complements BF and FD contain equal spaces.

For, since the diagonal AF bisects the rhomboid EI (I. 17. cor.), the triangle AEF is equal to AIF; and for the same reason the triangle FHC is equal to FGC, and likewise the whole triangle ABC is equal to ADC. From this triangle ABC on the one side of the diagonal, take away the two triangles AEF and FHC; and from the equal triangle ADC on the other side take away the two triangles AIF and FGC, and there remains the rhomboid BF equivalent to FD.

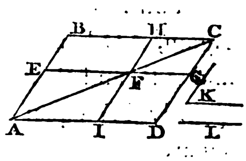



## PROP. IV. PROB.

With a given straight line to construct a rhomboid equivalent to a given rectilineal figure, and having an angle equal to a given angle.

Let it be required to construct a rhomboid, containing a given space, and having a side equal to the line L, and an angle equal to K.

Reduce the figure to a triangle (II. 2.), and let the rhomboid BF, with an angle BEF equal to K, have the altitude of the triangle, and EF equal to half its base; produce EF until FG be equal to L, through G draw DGC parallel to EB, and meeting the extension of BH in C; join CF, and produce it to meet the extension of BE in A; draw CD parallel to EF, meeting the extension of CG in D, and produce HF to meet AD to I: FD is the rhomboid required.

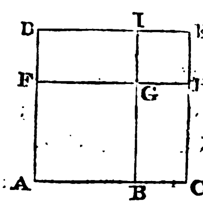



For FD and FB are evidently complementary rhomboids about the diagonal BC, and therefore (II. 3.) equivalent; and because AE and IF are parallel, the angle FID is equal to the interior angle EAI (I. 9. cor.), which again is equal to BEF or to the given angle K.

## PROP. V. THEOR.

The square constructed upon the sum of two straight lines, is equivalent to the squares of those lines, together with twice their rectangle.

If AB and BC be two straight lines placed continuous; the square ADEC erected upon their sum AC, is equivalent to the two squares of AB and BC, with twice the rectangle contained by them.




For, through B draw BI parallel to AD, make AF equal to AB, and through F draw FH parallel to AC or DE.

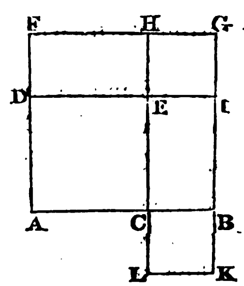
It is manifest that the spaces AG, GE, DG and CG, into which the square of AC is thus divided, are all rhom-

boidal and rectangular. And, because AB is equal to AF, and the opposite sides equal, the figure AG is equilateral, and having a right angle at A, is hence a square. Again, AD being equal to AC, take away the equals AF and AB, and there remains DF equal to BC, and consequently IG equal to GH: wherefore IH is likewise a square. The rectangle GD is contained by the sides FG and GI, which are equal to AB and BC; and the rectangle CG is contained by the sides GB and GE, which are likewise equal to AB and BC. Consequently the whole square of AC is compounded of the two squares of AB and BC, together with twice the rectangle contained by these lines.

## PROP. VI. THEOR.

The square constructed upon the difference of two straight lines, is equivalent to the squares of those lines, diminished by twice their rectangle.

Let AC be the difference of two straight lines AB and BC; the square of AC is equivalent to the excess of the two squares of AB and BC above twice their rectangle.




For let the squares of AB, and AC be completed, and that of BC formed below it; produce the sides CE and AE to H and I.

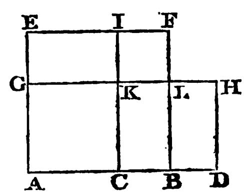
It is evident that GE is equal to BL, or the square of BC; to each of them add the intermediate rectangle EB, and GC is equal to IL; but the rectangle under AB and BC is equal to the rectangle

IL, which is also equal to DG. From the compound surface CAFGBKL, or the squares of AB and AC, take away the space DFGBKLC, or the rectangles IL and DG, that is, twice the rectangle under AB and BC,—and there remains ADEC, or the square of the difference AC of the two lines AB and BC.

## PROP. VII. THEOR.

The difference between the squares of two straight lines is equivalent to the rectangle contained by their sum and their difference.

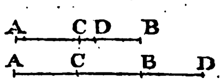
Let AB and BD be two continuous straight lines, from the greater of which is cut off a part BC equal to the extension BD; the excess of the square of AB above that of BD or BC is equivalent to the rectangle under their sum AD, and their difference AC.




For having made AG equal to AE, draw GH parallel to AD, and CI, DH both parallel to AE.

Because AG is equal to AC, the excess GK is equal to CB. But the opposite sides KI and GK are equal to AC, and IF and KL equal to CB; wherefore KF is the square of KL or BC, and EK, BK and BH are equal rectangles. To the rectangle BG join on the one side the rectangle EK, and on the other the rectangle BH, and the figure AEIKLB is equivalent to the rectangle AGHD, that is, the excess of the square of AB above that of BC is equivalent to the rectangle under AD and AC.

*Cor.* Hence, if a straight line AB be bisected in C, and cut unequally in D, either between the ends of AB, or in the extension of this line, the difference between the square of the half AC, and of the square of the interval CD between the points of section, will be equivalent to the rectangle under the segments AD and BD, whether considered as *internal* or *external.*



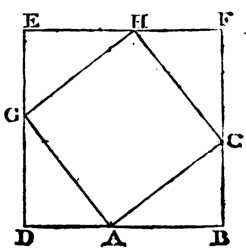

*General Corollary.* The preceding properties of the section of lines, all admit of arithmetical illustration; for a certain linear portion being adopted to measure length, and its square to measure surface, it is easy to perceive that lines being thus denoted by numbers, their squares will be expressed by the second powers, and their rectangles by the products of those numbers.

## PROP. VIII. THEOR.

The square constructed upon the hypotenuse of a right-angled triangle, is equivalent to the squares upon the two sides.

Let the triangle ABC be right angled at B; the square constructed on the hypotenuse AC is equivalent to the squares constructed on the base and perpendicular AB and BC.

For produce the base AB, till AD be equal to the perpendicular BC; upon the compound line DB describe the square BDEF, and make DG and EH equal to AB, and join AG, GH and HC.



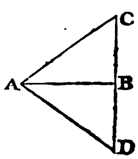

The triangles ABC and ADG, having the sides AB, BC evidently equal to DG, and AD, and the ight angle at B equal to that at

D, are (I. 3.) equal. In the same manner, the triangles HEG and CHF are proved to be equal to ABC. But (I. 9. cor.) the exterior angle GAB is equal to the interior angles ADG and AGD, from which take away the equal angles ABC and AGD, and there remains GAC equal to ADG, and consequently to a right angle. Wherefore the quadrilateral figure AGHC, having likewise all its sides equal, is a square. But (II. 5.) the square BDEF, described upon the sum of the sides AB and BC, is equivalent to the squares of those sides, together with twice their rectangle. Now the rectangle under AB and BC is double of the triangle ABC; and consequently the square BDEF is equivalent to the squares of AB and BC, and the four triangles CBA, ADG, GEH and HFC: but the same square is equivalent to the interior square AGHC, with those four triangles; wherefore the squares of the base AD, and of the perpendicular BC, are equivalent to the single square described on the hypotenuse AC.

## PROP. IX. THEOR.

If the square of one side of a triangle be equivalent to the squares of both the other sides, that side subtends a right angle.

Let the square described on AC be equivalent to the two squares of AB and BC; the triangle ABC is right-angled at B.




For draw BD perpendicular to AB (I. 20.) and equal to BC, and join AD.

Because BC is equal to BD, the square of BC is equal to the square of BD, and consequently the squares of AB and BC are equal to the

squares of AB and BD. But the squares of AB and BC are, by hypothesis, equivalent to the square of AC; and since ABD is, by construction, a right angle, the squares of AB and BD are, by the preceding proposition, equivalent to the square of AD. Whence the square of BC is equal to that of AD, and the line BC itself equal to AD. The two triangles ACB and ADB having all the sides in the one respectively equal to those in the other, are therefore equal (I. 2.), and consequently the angle ABC is equal to the corresponding angle ABD, that is, to a right angle.

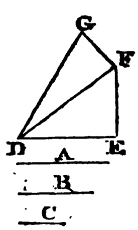
*A right angle of cord is hence made for field operations, by placing the knots at the intervals of* 3, 4, *and* 5 *measures.*

## PROP. X. PROB.

To find the side of a square equivalent to any number of given squares.

Let A, B, and C be the sides of the squares, to which it is required to find an equivalent square.

Draw DE equal to A, and from its extremity E erect (I. 34.) the perpendicular EF equal to B, join DF, and again, perpendicular to this, draw FG equal to C, and join DG: DG is the side of the square which was required.




For since DEF is a right-angled triangle, the square of DF is equivalent to the squares of DE and EF (II. 8.) or of the lines A and B. Add on both sides the square of FG or of C, and the squares of DF and FG, which are equivalent to the square of DG (II. 10.), are equivalent to the aggregate

squares of A, B, and C. And, by thus repeating the process, it may be extended to any number of squares.

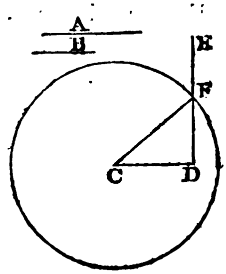
*By help of this proposition, the square root of a number can be extracted geometrically; for every number is resolvable into squares, not exceeding four.*

## PROP. XI. PROB.

To find the side of a square equivalent to the difference between two given squares.

Let A and B be the sides of two squares; it is required to find a square equivalent to their difference.

Draw CD equal to the smaller line B, from its extremity erect (I. 20.) the indefinite perpendicular DE, and about the centre C, with a distance equal to the greater line A, describe a circle cutting DE in F: DF is the side of the square required.




For join CF. The triangle CDF being right-angled, the square of its hypotenuse CF is equivalent to the squares of CD and DF (II. 8.), and consequently taking the square of CD from both, the excess of the square of CF above that of CD is equivalent to the square of DF, or the square of DF is equivalent to the excess of the square of the line A above that of B.

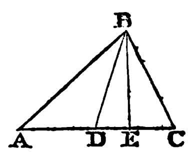
Since by Proposition II. 7, the difference between the squares of two lines is equivalent to the rectangle under their sum and difference, if A and B be taken equal to half the sum and half the difference of a given rectangle, its equivalent square may be hence found.

## PROP. XII. THEOR.

The difference between the squares of the sides of a triangle, is equivalent to twice the rectangle contained by the base and the distance of its middle point from the perpendicular.

Let the side AB of the triangle ABC be greater than BC; and, having let fall the perpendicular BE, and bisected AC in B, the excess of the square of EB above that of BC is equivalent to twice the rectangle contained by the base AC and the segment DE.

For the square of AB is equivalent to the squares of DE and BE (II. 8.), and the square of BC is equivalent to the squares CE and BE; wherefore, since the square of BE occurs in both, the excess of the square of AB above that of AC is equivalent to the excess of the square of AE above that of CE. But the excess of the square of DE above that of CE is (II. 7.) equivalent to the rectangle contained by their sum BC and their difference, which is evidently the double of DE, the distance of the point E from the middle D; and consequently the difference between the squares of AE and CE, being equivalent to the rectangle contained by AC and the double of DE, is equivalent to twice the rectangle under AC and DE.



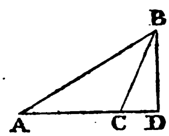

*Cor.* Hence the difference of the squares of the sides of a triangle is equivalent to the difference of the squares of the segments of the base, made by a perpendicular from the vertex.

*By the application of this or the succeeding proposition, the three sides of a triangle being numerically given, the perpendicular let fall from the vertex upon the base is formed, and thence, by Proposition II. 1, the area of the triangle is computed.*

## PROP. XIII. THEOR.

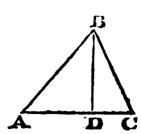
**The square of the side of a triangle is greater or less than the squares of the base and the other side, according as the opposite angle is obtuse or acute, by twice the rectangle contained by the base and the distance intercepted between the vertex of that angle and the perpendicular.**

In the obtuse-angled triangle ABC, where the perpendicular BD falls without the base; the square of the side AB which subtends the obtuse angle exceeds the squares of the sides AC and BC which contain it, by twice the rectangle under AC and CD.




For the square of AD, or the square of the sum of AC and CD is (II. 5.) equivalent to the squares of these lines AC, CD, together with twice their rectangle. Add to both the square of BD, and the squares of AD, BD, or (II. 8.) the square of AB is equivalent to the squares of CD, BD, together with twice the rectangle BC, CD; but the squares of CD, BD are (II. 8.) equivalent to the square of BC; whence the square of AB exceeds the squares of AC, BC, by twice the rectangle under AC and CD.

Again, in the acute-angled triangle ABC, where the perpendicular BD falls within the triangle; the square of the side AB which subtends the acute angle is less than the squares of the containing sides AC, BC, by twice the rectangle under the base AC and its intercepted portion CD.




For the square of AD, or the square of the difference between AC and CD (II. 16.), is equivalent to their

squares, diminished by twice their rectangle. Add to each of these equalities the square of BD, and the squares of AD and BD, or the square of AB (II. 8.), are equivalent to the square of BD, with the squares of AC and CD, or (II. 8.) to the square of BC, diminished by twice the rectangle under AC and CD. Consequently the square of AB is less than the squares of AC and BC, by twice the rectangle under AC and CD.

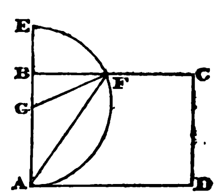
*Cor.* If the triangle ABC be isosceles, having equal sides AC and BC, the square of the base AB is equivalent to twice the rectangle under the side BC, and the adjacent segment CD made by the perpendicular BC, whether the vertical angle be obtuse or acute. For the square of AB is equivalent to the squares of AC and BC, or to twice the square of BC increased or diminished by twice the rectangle under BC and CD; that is, equivalent to twice the rectangle under BC and CD, the sum or difference of BC and CD.

## PROP. XIV. PROB.

To find the side of a square equivalent to a given rectangle.

Let it be required to find the side of a square equivalent to the rectangle ABCD.

Produce AB, the shorter side of the rectangle till AE be equal to the longer side AD, and upon it describe a semicircle and cutting BC in F; AF being joined, is the side of the equivalent square.




For FG being drawn to the centre, the triangle AGF is isosceles; and therefore, (by the last corollary,) the square of AF is

equivalent to twice the rectangle under AB and AG, or to the rectangle under AB, and the double of AG or AD.

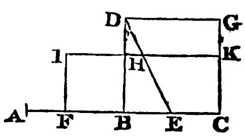
A perpendicular let fall from D upon AF will form with AD and the adjacent part of AF a triangle equal to AEF, containing the chord EF; and the square on that perpendicular being completed, is easily distinguished into three portions, which recombine into the rectangle ABCD.

*Hence the square root of any composite number may be determined geometrically, the factors denoting the sides of the rectangle.*

## PROP. XV. PROB.

**To divide a given straight line, so that the square of one part shall be equivalent to the rectangle contained by the whole line and its remainder.**

Let AB be the straight line which it is required to cut in two segments, BF and AF, such that the square of the former shall be equivalent to the rectangle contained by AB and the latter.




Produce AB till BC be equal to it, erect the perpendicular BD equal to AB or BC, bisect BC in E, join ED and make EF equal to it; the square of the segment BF is equivalent to the rectangle contained by the whole line BA and its remaining segment AF.

For complete the square BG, make BH equal to BF, and draw IHK and FI parallel to AC and BD. Since AB is equal to BD, and BF to BH; the remainder AF is equal to HD: and it is farther evident, that FH is a square, and that IC and DK are rectangles. Now BC being bisected in E and cut externally in F, the rectangle CF, FB, or the figure IC, together with the square of BE

is (II. 7. cor.) equivalent to the square of EF or of DE. But the square of the hypotenuse DE is equivalent to the squares of DB and BE (II. 8.); whence the rectangle IC, with the square of BE, is equivalent to the squares of DB and BE; or, omitting the common square of BE, the rectangle IC is equivalent to the square of DB. Take away from both the rectangle BK, and there remains the square BI, or the square of BF, equivalent to the rectangle HG, or the rectangle contained by BA and AF.

*Cor.* Since the rectangle under CF and FB is equivalent to the square of BC, it is evident that the line CF is likewise divided at B in the same manner as the original line AB. But the line CF is made up, by joining the whole line AB, now become only the larger portion, to its greater segment BF, which next forms the smaller portion in the new compound. Hence this peculiar division of any line being once obtained, a series of other lines, all possessing the same property, may readily be found, by repeated additions. Thus, let AB be so cut, that the square of BC is equivalent to the rectangle BA, AC: Make, successively, BD equal to BA, DE equal to DC,


EF equal to EB, and FG equal to FD; the lines CD, BE, DF, and EG, beginning at the successive points C, B, D, and E are divided at the points B, D, E, and F, such that, in each of them, the square of the larger part is equivalent to the rectangle contained by the whole and the smaller part.

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*It will be convenient, for the sake of conciseness, to designate in future this remarkable division of a line, where the rectangle under the whole and one part is equivalent to the square of the other, by the term* Medial Section.

# RUDIMENTS

## OF

# PLANE GEOMETRY.

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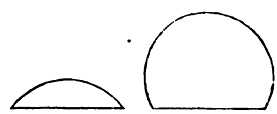
## BOOK III.

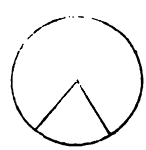
### DEFINITIONS.

1. Any portion of the circumference of a circle is called an *arc*, and the straight line which joins the two extremities, a *chord*.

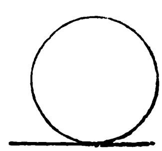


2. The space included between an arc and its chord, is named a *segment*.

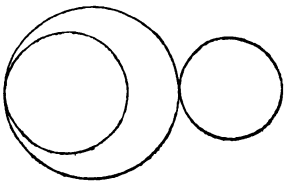
3. A *sector* is the portion of a circle contained by two radii and the arc lying between them.



4. The *tangent* to a circle is a straight line which *touches* the circumference, or meets it in a single point.



5. Circles are said to *touch* mutually, if they meet, but do not cut each other.



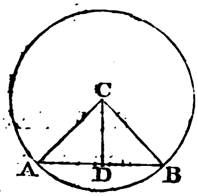
6. The point where a straight line touches a circle, or one circle touches another, is called the point of *contact*.

## PROP. I. THEOR.

The perpendicular from the centre of a circle upon a chord, bisects it.

The perpendicular drawn from the centre C to the chord AB, cuts it into two equal parts AD, DB.

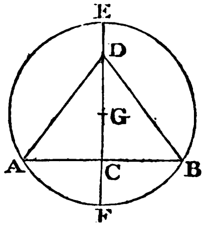
For join CA, CB: In the triangles ACD, BCD, the side AC is equal to CB; CD is common to both, and the right angle ADC is equal to BDC; these triangles having also their corresponding angles at A and B both acute, are equal (I. 13. & 14. cor.) and consequently the side AD is equal to BD.




## PROP. II. THEOR.

The straight line which bisects a chord at right angles, passes through the centre of the circle.

If the perpendicular FE bisect the chord AB, it will pass through G the centre of the circle.

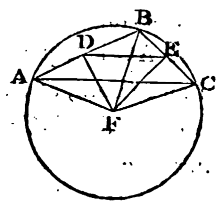



For in FE take any point D, and join DA and DB. The triangles ADC and BDC, having the side AC equal to BC, the right angle ACD equal to ACD, and the side CD common, are equal (I. 3.), and consequently the base AD is equal to BD. The centres of all the circles that can pass through A and B thus range in EF, and hence the centre G of the circle AEBF must hence occur in that perpendicular.

## PROP. III. PROB.

### To describe a circle about a given triangle.

Let it be required to describe a circle about the triangle ABC.



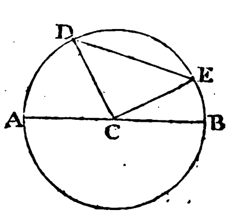

Bisect the side AB by the perpendicular DF, and BC by the perpendicular EF. These straight lines DF, EF will meet; DE being joined, the angles EDF, DEF are less than BDF, BEF, and consequently less than two right angles; and DF, EF must concur to form a triangle whose vertex is F. This point is therefore (II. 3.) the centre of a circle which passes through the points A and B, and through B and C, and consequently circumscribes the triangle.

*Cor.* Hence an arc being given, the circle to which it belongs may be completed.

## PROP. IV. THEOR.

### The diameter is the greatest chord in a circle.

The diameter AB is greater than any other chord DE.



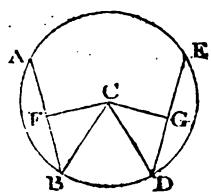

For join CD and CE. The two sides DC and EC of the triangle DCE are together greater than the third side DE (I. 15.): But DC and CE are equal to AC and CB, or to the whole diameter AB; which is therefore greater than DE.

## PROP. V. THEOR.

Equal chords are equidistant from the centre of a circle.

Let AB, DE be equal chords inflected within the same circle; their distances from the centre, or the perpendiculars CF, CG, let fall upon them, are equal.

For the perpendiculars CF and CG bisect the chords AB and DE (III. 1.), and consequently their halves BF, DG, are equal. The right-angled triangles CBF and CDG, which are likewise of the same character, having the two sides BC, BE equal respectively to DC, DG, and the corresponding angle BFC equal to DGC, are equal (I. 14. cor.), and consequently the side FC is equal to GC.



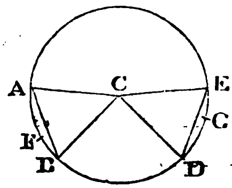

## PROP. VI. THEOR.

In the same or equal circles, equal angles at the centre are subtended by equal chords, and terminated by equal arcs.

If the angle ACB at the centre be equal to DCE, the chord AB is equal to DE, and the arc AFB equal to DGE.

For let the sector ACB be applied to DCE. The centre remaining in its place, the radius CA will lie on CD: and the angle ACB being equal to DCE, the radius CB

will adapt itself to CE. And because all the radii are equal, their extreme points A and B must coincide with D and E; wherefore the straight lines which join those points, or the chords AB and DE, must coincide. But the arcs AFB and DGE that connect the same points, will also coincide; for any intermediate point F in the one, being at the same distance from the centre as every point of the other, must, on its application, find always a corresponding point G.




The same mode of reasoning is applicable to the case of equal circles.

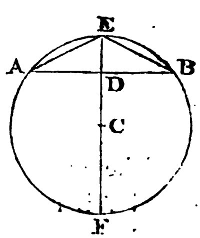
## PROP. VII. THEOR.

To bisect a given arc of a circle.

Let it be required to divide the arc AEB into two equal portions.

Draw the chord AB, and (I. 19. cor.) bisect it by the perpendicular EF cutting the circumference AB in E: The arc AE is equal to EB.

For the triangles ADE, BDE, have the side AD equal to BD, the side DE common, and the containing right angle ADE equal to BDE; they are (I. 3.) consequently equal, and have the base AE equal to BE. But these equal chords AE, BE must subtend equal arcs of a like kind, and the arcs AE, BE are evidently each of them less than a semicircumference.




It is obvious that the diameter EDF bisects likewise the opposite arc AFB.

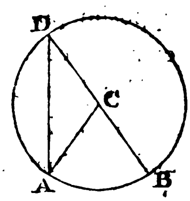
## PROP. VIII. THEOR.

**The angle at the centre of a circle is double of the angle which, standing on the same arc, has its vertex in the circumference.**

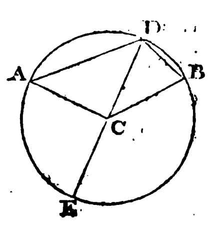
Let AB be an arc of a circle; the angle which it terminates at the centre is double of ADB, the corresponding angle at the circumference.

For join DC and produce it to the opposite circumference. This diameter DCE, if it lie not on one of the sides of the angle ADB, must either fall within that angle or without it.

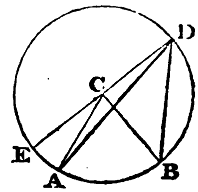
First, let DC coincide with DB. And because AC is equal to DC, the angle ADC is equal to DAC (I. 6.); but the exterior angle ACB is equal to both of these (I. 11.), and therefore equal to double of either, or the angle ACB at the centre is double of the angle ADB at the circumference.




Next, let the straight line DCE lie within the angle ADB. From what has been demonstrated, it is apparent, that the angle ACE is double of ADE, and the angle BCE double of BDE; wherefore the sum of the angles ACE, BCE, or the whole reverse angle ACB, is double of that of the angles ADE, BDE, or the compound angle ADB at the circumference.



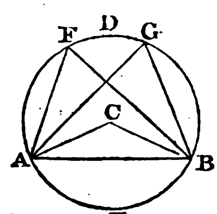
Lastly, let DCE fall without the angle ADB. Because the angle BCE is double of BDE, and the angle ACE is double of ADE; the excess of BCE above ACE, or the angle ACB at the centre, is double of the excess of BDE above ADE, that is, of the angle ADB at the circumference.

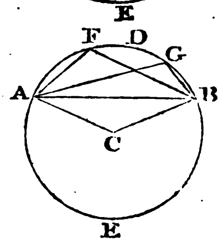



## PROP. IX. THEOR.

The angles in the same segment of a circle are equal.

Let ADB be the segment of a circle; the angles AFB, AGB contained in it, or which stand on the same opposite portion AEB of the circumference, are equal to each other.




For join CA, CB. The angle ACB, or its reverse at the centre, and terminated by the arc AEB, is double of the angle AFB or AGB at the circumference (III. 8.); these angles AFB, AGB, which stand on the same arc AEB, are, therefore, in every case, the halves of the same central angle ACB, and are consequently equal to each other.

*Cor.* Hence equal angles at the circumference must stand on equal arcs; for their doubles or the central angles, being equal, are terminated by equal arcs.

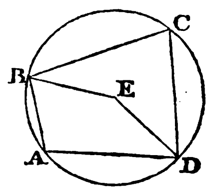
***Hence the mechanical way of tracing the arcs of a circle without the help of compasses.***

## PROP. X. THEOR.

The opposite angles of a quadrilateral figure contained within a circle, are together equal to two right angles.

Let ABCD be a quadrilateral figure inscribed in a circle; the angles at A and C are together equal to two right angles, and so are those at B and D.

For join EB and ED. The angle BED at the centre is double of the angle BCD at the circumference (III. 8.); and for the same reason, the reverse angle BED is double of BAD. Consequently the angles BCD and BAD are the halves of angles about the point E, which make up four right angles; wherefore the angles BCD and BAD are together equal to two right angles.




In the same manner, by joining EA and EC, it may be proved that the angles ABC and ADC are together equal to two right angles.

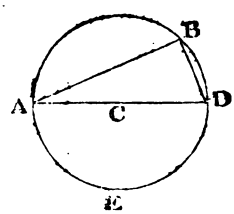
*Cor.* Hence the exterior angle of a quadrilateral figure inscribed in a circle is equal to the opposite angle; and hence the angles at the base of a triangle inscribed in a circle, are together equal to the angle contained in the segment opposite to its vertex.

*This proposition marks the condition required to make a circle pass through four given points.*

## PROP. XI. THEOR.

**The angle in a semicircle is a right angle, the angle in a greater segment is acute, and the angle in a smaller segment is obtuse.**

Let ABD be an angle in a semicircle, or standing on the circumference AED; it is a right angle.




For ABD, being an angle at the circumference, is half of the angle at the centre terminated by the same arc AED (III. 8.); it is, therefore, half of the angle ACD formed by the diverging of the opposite portions CA, CD of the diameter, or half of two right angles, and is consequently equal to one right angle.

*Cor.* Hence a simple method of drawing perpendiculars, yet capable of various applications.

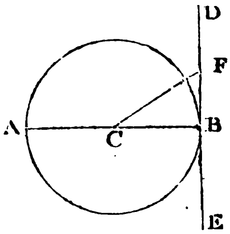
*A perpendicular may be traced on the ground by means of an extended cord, to the middle of which is tied another cord of half the length.*

## PROP. XII. THEOR.

**The perpendicular at the extremity of a diameter is a tangent to the circle.**

Let ACB be the diameter of a circle, to which the straight line EBD is drawn at right angles from the ex-

tremity B; it will touch the circumference at that point.




For from the centre draw CF to any other point in BD. The right angle CBF being greater than the acute angle CFB, the opposite side CF (I. 7.) must be greater than BC. Every point in BD, except only B, lies thus without the circle.

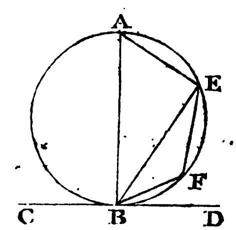
*Cor.* Hence the perpendicular to a tangent at the point of contact, must pass through the centre of the circle.

If an extended line were to turn about B, it might cut the circumference first above the diameter AB, and next below it; but it would merge in a tangent in that limit of transition, when it cuts neither above nor below.

## PROP. XIII. THEOR.

If, from the point of contact, a straight line be drawn to cut the circumference, the angles which it makes with the tangent are equal to those in the alternate segments of the circle.

Let CD be a tangent, and BE a straight line drawn from the point of contact, cutting the circle into two segments BAE and BFE; the angle EBD is equal to EAB, and the angle EBC to EFB.




For draw BA perpendicular to CD, join AE, and from any point F in the opposite arc, draw FB and FE.

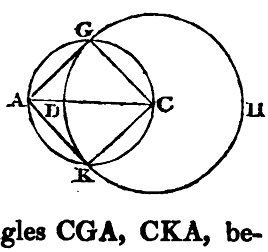
Because BA is perpendicular to the tangent at B, it is a diameter (III. 12. cor.), and AEFB a semicircle; wherefore (III. 11.) AEB is a right angle, and the remaining acute angles BAE, ABE of the triangle, being (I. 11.) together equal to another right angle, are equal to ABE and EBD, which compose the right angle ABD. Take the angle ABE away from both, and the angle BAE remains equal to EBD.

Again, the opposite angles BAE and BFE of the quadrilateral figure BAEF, being equal to two right angles (III. 10.), are equal to the angle EBD with its adjacent angle EBC; and taking away the equals BAE and EBD, there remains the angle BFE equal to EBC.

## PROP. XIV. THEOR.

To draw a tangent to a circle, from a given point without it.

Let A be a given point, from which it is required to draw a straight line that shall touch the circle DGH.




Join AC with the centre, and on this as a diameter describe the circle AGCK, cutting the given circle in the points G, K: join AG, AK; either of these lines is the tangent required.

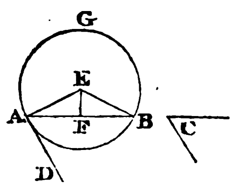
For join CG, CK. And the angles CGA, CKA, being each in a semicircle, are right angles (III. 11.), and consequently AG, AK, touch the circle DGHK at the points G, K (III. 12.).

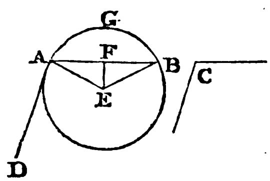
## PROP. XV. THEOR.

**On a given straight line, to describe a segment of a circle that shall contain an angle equal to a given angle.**

Let AB be a straight line, on which it is required to describe a segment of a circle containing an angle equal to C.

If C be a right angle, it is evident that the problem will be performed, by describing a semicircle on AB. But if the angle C be either acute or obtuse; draw AD making an angle BAD equal to C, erect AE perpendicular to AD, draw EF to bisect AB at right angles and meeting AE in E, and from this point as a centre and with the distance EA, describe the required segment AGB.




Because EF bisects AB at right angles, the circle described through A must also pass through (III. 5.) the point B; and since EAD is a right angle, AD touches the circle at A (III. 12.), and the angle BAD, which was made equal to C, is equal (III. 13.) to the angle in the alternate segment AGB.

*By this proposition, the position of a point is determined from the angles observed in reference to three given points.*

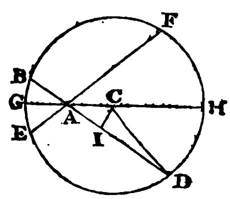
## PROP. XVI. THEOR.

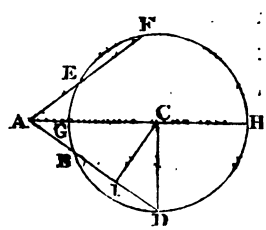
**If through a point, within or without a circle, two straight lines be drawn to cut the circumference; the rectangle under the segments of the one, is equivalent to that contained by the segments of the other.**

Let the two straight lines AD and AF be extended through the point A, to cut the circumference BFD of a circle; the rectangle contained by the segments AE and AF of the one, is equivalent to the rectangle under AB and AD, the distances intercepted from A in the other.

For draw AC to the centre, and produce it both ways to terminate in the circumference at G and H; let fall the perpendicular CI upon BD (I. 5.), and join CD.

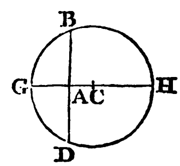
Because CI is perpendicular to AD, the difference between the squares of CA and CD, the sides of the triangle ACD is equivalent to the difference between the squares of the segments AI and ID the segments of the base (II. 12. cor.); and the difference between the squares of two straight lines being equivalent to the rectangle under their sum and their difference (II. 7.), the rectangle contained by the sum and difference of AC, CD is equivalent to the rectangle contained by the sum and difference of AI, ID. But since the radius CG is equal to CH, the sum of AC and CD is AH, and their



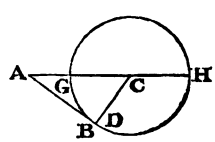



difference is AG; and because the perpendicular CI bisects the chord BD (III. 2.), the sum of AI and ID is AD, and their difference AB. Wherefore the rectangle AH, AG is equivalent to the rectangle AB, AD. In the same way it is proved, that the rectangle AH, AG is equivalent to the rectangle AE, AF; and consequently the rectangle AE, AF, is equivalent to the rectangle AB, AD.

*Cor.* 1. If the vertex A of the straight lines lie within the circle and the point I coincide with it, BD, being then at right angles to CA, is bisected at A (III. 2.), and the rectangle AB, AD becomes the same as the square of AB. Consequently the square of any perpendicular AB limited by the circumference is equivalent to the rectangle under the segments AG, AH, into which it divides the diameter.




*Cor.* 2. If the vertex A lie without the circle and the point I coincide with B or D, the angle ABC being then a right angle, the incident line AB must be a tangent (III. 12.), and consequently the two points of section B and D coalesce in a single point of contact. Wherefore the rectangle under the distances AB, AD becomes the same as the square of AB; and consequently the rectangle contained by the segments AG, AH of the diameter, is equivalent to the square of the tangent AB.




By the application of either of these corollaries, the side of a square may be found equivalent to a given rectangle.

# RUDIMENTS

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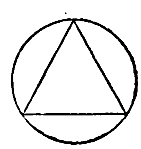
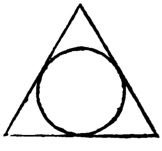
## BOOK IV.

### DEFINITIONS.

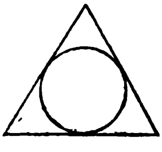
1. A rectilineal figure is said to be *inscribed* in a circle, when all its angular points lie on the circumference.

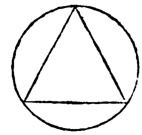


2. A rectilineal figure *circumscribes* a circle, when each of its sides is a tangent.



3. A circle is *inscribed* in a rectilineal figure, when it touches all the sides.





4. A circle is *described* about a rectilineal figure or *circumscribes* it, when the circumference passes through all the angular points of the figure.

5. Polygons are *equilateral*, when their sides, in the same order, are respectively equal: They are *equiangular*, if an equality obtains between their correspondiug angles.

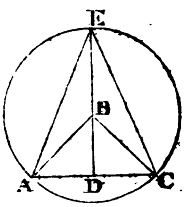
6. Polygons are said to be *regular*, when all their sides and their angles are equal.

7. A figure of *five* angles or sides is called a *pentagon*; a *six-sided* figure, a *hexagon*; an *eight-sided* figure, an *octagon*; a *ten-sided* figure, a *decagon*; and a *twelve-sided* figure, a *dodecagon*.

## PROP. I. PROB.

An isosceles triangle being given, to construct another on the same base, but with only half the vertical angle.

Let ABC be an isosceles triangle standing on AC; it is required, on the same base, to construct another isosceles triangle, that shall have its vertical angle equal to half of the angle ABC.

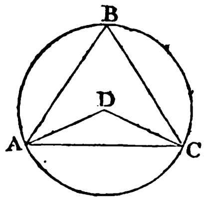



Bisect AC in D (I. 19.), join DB, which produce till BE be equal to BA or BC, and join AE, CE: AEC is the isosceles triangle required.

For, the straight line BE being equal to BA and BC, the point B is the centre of a circle which passes through the points A, E, and C; and consequently the angle ABC is the double of AEC at the circumference (III. 8.), or the vertical angle AEC is half of ABC. But the triangles AED and CED, having the side DA equal to DC, the side DE common to both, and the right angle ADE (III. 2.) equal to CDE, are (I. 14. cor.) equal, and consequently AE is equal to CE. Wherefore the triangle AEC is likewise isosceles.

## PROP. II. PROB.

**Given an acute-angled isosceles triangle, to construct another on the same base, which shall have double the vertical angle.**




Let ABC be an acute-angled isosceles triangle; it is required, on the base AC, to construct another isosceles triangle, having its vertical angle double of the angle ABC.

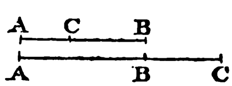
Describe a circle through the three points A, B, and C (III. 3. cor.), and draw AD, CD to the centre D; the triangle ADC is the isosceles triangle required. For the angle ADC, being at the centre of the circle, is (III. 8.) double of ABC, the corresponding angle at the circumference.

## PROP. III. PROB.

**Given either the base or one of the sides, to construct an isosceles triangle, which shall have each of the angles at its base a double vertical angle.**

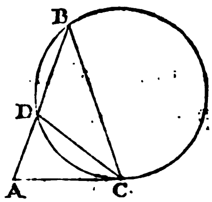
When the side AB is given, divide it by a medial section at C, and the greater segment will express the base of the isosceles triangle.

If the base AB be given, produce it till BC be equal to the greater segment of its medial section, and the compound line AC will represent the side of the isosceles triangle.



It only remains to show that a triangle so formed will fulfil the required conditions.

Let D be the medial section of the side AB, join CD, and (III. 3.) about the triangle BDC describe a circle.




Since the rectangle AB, AD is by construction equivalent to the square of BD or AC, the line AC must (III. 16. cor. 2.) terminate in the circumference of the circle, and is hence a tangent. Wherefore (III. 18.) the angle ACD is equal to the alternate angle CBD; add DCB to each, and the angles ACD and DCB, making the whole angle ACB, are equal to CBD and DCB, and consequently (I. 11.) to the exterior angle ADC. The angle ACB or CAD is thus equal to ADC, and hence CD (I. 6.) is equal to AC or BD. The triangle BDC is therefore isosceles, and the angle CBD equal to BCD or DCA. Whence the whole angle BCA at the base of the triangle is double of the vertical angle ABC.

It is obvious, from this demonstration, that the small interior triangle ACD is likewise isosceles, and with the same distinguishing property, of having the angles at its base each double of the vertical angle.

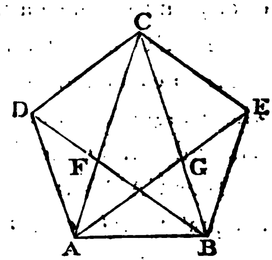
*Cor.* Hence in such isosceles triangles the vertical angle is equal to the fifth part of two right angles; for the angles at the base being each of them double of the vertical angle, they are both equal to four times it, and consequently this vertical angle must be the fifth part of all the angles of the triangle, or of two right angles.

## PROP. IV. PROB.

**On a given straight line, to construct a regular pentagon.**

Let AB be the straight line, on which it is required to form a regular five-sided figure.

On AB erect (IV. 4.) the isosceles triangle ACB, having each of the angles at its base double its vertical angle, from A as a centre with the distance AC describe an arc of a circle, and from B as a centre with the same distance describe another arc, and from the vertex C inflect the straight lines CE, CD each equal to AB: Join AD and BE, and the figure ADCEB is the pentagon required.




For the intersecting isosceles triangles ACB, CBD, and CAE are evidently equal, and have their base angles CAB, CBA, CDB, DCB, ACE or AEC double of the vertical angles ACB, CBD or CAE. Again, the segments AF, FD, and BG, GE being equal, the obtuse triangles AFD and BGE are isosceles, and so likewise are the large obtuse triangles BAD and ABE. All the sides, therefore, of the figure are mutually equal, and they contain angles which are each triple of the vertical angle ACB of the elemental triangle.

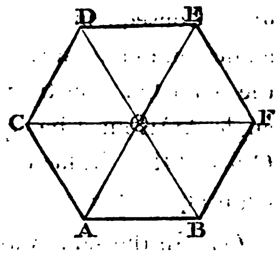
## PROP. V. PROB.

On a given straight line, to construct a regular hexagon.

Let AB be the given straight line, on which it is required to form a regular six-sided figure.

On AB erect the equilateral triangle AOB, and repeat equal triangles about the vertex O; these will together compose the hexagon required.

Because AOB is an equilateral triangle, each of its angles is equal to the third part of two right angles; wherefore the vertical angle AOB is the sixth part of four right angles, or six of such angles may be placed about the point O. But the bases of the triangles AOB, AOC, COD, DOE, EOF, and



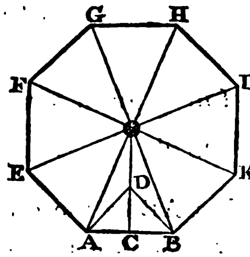

BOF are all equal; and so are the angles at the bases, which, taken by pairs, form the internal angles of the figure BACDEF. This figure is, therefore, a regular hexagon.

## PROP. VI. PROB.

On a given straight line, to describe a regular octagon.

Let AB be the given straight line, on which it is required to describe a regular eight-sided figure.

Bisect AB (I. 19.) by the perpendicular CD, which make equal to CA or CB, join DA and DB, produce CD until DO be equal to DA or DB, draw AO and BO, thus forming (IV. 1.) an angle equal to the half of ADB; and about the vertex O, repeat the equal triangles AOB, AOE, EOF, FOG, GOH, HOI, IOK, and join KB, to compose the octagon.




For the distances AD, BD are evidently equal; and because CA, CD, and CB are all equal, the angle ADB is contained in a semicircle, and is therefore a right angle. Consequently AOB is equal to the half of a right angle, and eight such angles will adapt themselves about the point O: thus the vertical angle of the succeeding triangle AOE must be equal to AOB, and likewise the side OE to OB. Whence the figure BAEFGHIK, having eight equal sides and equal angles, is a regular octagon.

*Cor.* In the same way, the sides of any polygon may be always doubled.

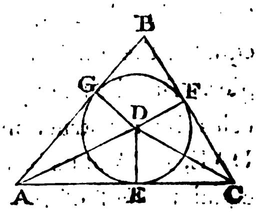
## PROP. VII. PROB.

In a given triangle, to inscribe a circle.

Let ABC be a triangle, in which it is required to inscribe a circle.

Draw AD and CD to bisect the angles CAB and ACB, and from their point of concourse D, with its distance DE from the base, describe the circle EFG: This circle will touch the triangle internally.

For (I. 5.) let fall the perpendiculars DG and DF upon the sides AB and BC. The triangles ADE, ADG, having the angle DAE equal to DAG, the right angle DEA equal to DGA, and the interjacent side AD common, are equal (I. 14. cor.), and therefore the side DE is equal to DG. In the same manner, it is proved, from the equality of the triangles CDE, CDF that DE is equal to DF; consequently DG is equal to DF, and the circle passes through the three points E, G, and F. But it also touches (III. 12.) the sides of the triangle in those points, for the angles DEA, DGA, and DFC, are all of them right angles.

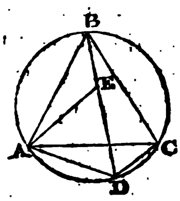



## PROP. VIII. THEOR.

A straight line drawn from the vertex of an equilateral triangle inscribed in a circle to any point in the opposite circumference, is equal to the two chords inflected from the same point to the extremities of the base.

Let ABC be an equilateral triangle inscribed in a circle, and BD, AD, and CD chords drawn from its three corners to a point D in the circumference; BD, which crosses the base AC, is equal to AD and CD taken together.

For, from BD cut off DE equal to DA, and join AE.




The angle ADB is equal to ACB (III. 9.) in the same segment, which, being the angle of an equilateral triangle, is equal to the third part of two right angles. But the triangle ADE being isosceles by construction, has the angles DAE, DEA at its base equal, and each of them, therefore, equal to half of the remaining two-thirds of two right angles, or equal to one-third part. Consequently ADE is equilateral, and its angle DAE equal to CAB; take CAE from both, and there remains the angle DAC equal to EAB; but the angle ABD is equal to ACD in the same segment. And thus the triangles ADC and AEB have the angles DAC, DCA equal to EAB, EBA, and the interjacent side AC equal to AB; they are consequently equal, and the side DC is equal to EB. But DE was made equal to DA; wherefore DA and DC are together equal to DE and EB, or to DB.

## PROP. IX. PROB.

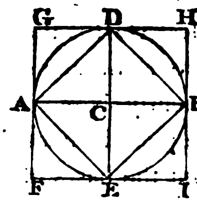
In and about a given circle, to inscribe and circumscribe a square.

Let EADB be a circle in which it is required to inscribe a square.

Draw the diameter AB, cross it (I. 4. cor.) by the perpendicular ED through the centre, and join AD, DB, BE, and EA: The inscribed figure ADBE is a square.

The angles about the centre C, being right angles, are equal to each other, and are, therefore, subtended by equal

chords, AD, DB, BE, and AE, but one of the angles ADB, being in a simicircle, is (III. 11.) a right angle, and consequently ADBE is a square.




Next, let it be required to circumscribe a square about the circle.

Apply tangents FG, GH, HI, and FI at the extremities of the perpendicular diameters: These will form a square.

For all the angles of the quadrilateral figure CG, being together equal to four right angles, and those at C, A, and D being each a right angle, the remaining angle at G is also a right angle, and CG is a rectangle; but it is likewise a square, since AC is equal to CD. In the same manner, CH, CI, and CF are proved to be squares; the sides FG, GH, HI, and IF of the exterior figure, being therefore the doubles of equal lines, are mutually equal, and the angle at G being a right angle, FH is consequently a square.

*Cor.* Hence the circumscribing square is double of the inscribed square, and this again is double of the square constructed on the radius of the circle.

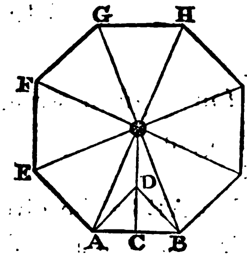
## PROP. X. PROB.

To inscribe a regular hexagon in a given circle.

Let it be required, in the circle FBD, to inscribe a hexagon.

Draw the radius OA, on which construct the equilateral triangle ABO, and repeat the equal triangles about the vertex O: These triangles will compose a hexagon.

Bisect AB (I. 19.) by the perpendicular CD, which make equal to CA or CB, join DA and DB, produce CD until DO be equal to DA or DB, draw AO and BO, thus forming (IV. 1.) an angle equal to the half of ADB; and about the vertex O, repeat the equal triangles AOB, AOE, EOF, FOG, GOH, HOI, IOK, and join KB, to compose the octagon.




For the distances AD, BD are evidently equal; and because CA, CD, and CB are all equal, the angle ADB is contained in a semicircle, and is therefore a right angle. Consequently AOB is equal to the half of a right angle, and eight such angles will adapt themselves about the point O: thus the vertical angle of the succeeding triangle AOE must be equal to AOB, and likewise the side OE to OB. Whence the figure BAEFGHIK, having eight equal sides and equal angles, is a regular octagon.

*Cor.* In the same way, the sides of any polygon may be always doubled.

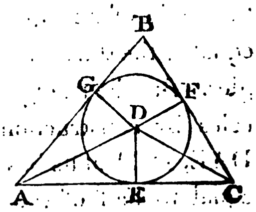
## PROP. VII. PROB.

In a given triangle, to inscribe a circle.

Let ABC be a triangle, in which it is required to inscribe a circle.

Draw AD and CD to bisect the angles CAB and ACB, and from their point of concourse D, with its distance DE from the base, describe the circle EFG: This circle will touch the triangle internally.

For (I. 5.) let fall the perpendiculars DG and DF upon the sides AB and BC. The triangles ADE, ADG, having the angle DAE equal to DAG, the right angle DEA equal to DGA, and the interjacent side AD common, are equal (I. 14. cor.), and therefore the side DE is equal to DG. In the same manner, it is proved, from the equality of the triangles CDE, CDF that DE is equal to DF; consequently DG is equal to DF, and the circle passes through the three points E, G, and F. But it also touches (III. 12.) the sides of the triangle in those points, for the angles DEA, DGA, and DFC, are all of them right angles.



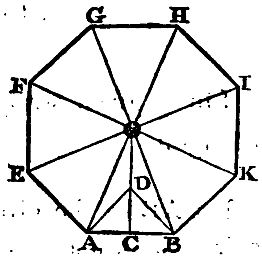

## PROP. VIII. THEOR.

A straight line drawn from the vertex of an equilateral triangle inscribed in a circle to any point in the opposite circumference, is equal to the two chords inflected from the same point to the extremities of the base.

Let ABC be an equilateral triangle inscribed in a circle, and BD, AD, and CD chords drawn from its three corners to a point D in the circumference; BD, which crosses the base AC, is equal to AD and CD taken together.

For, from BD cut off DE equal to DA, and join AE.

Bisect AB (I. 19.) by the perpendicular CD, which make equal to CA or CB, join DA and DB, produce CD until DO be equal to DA or DB, draw AO and BO, thus forming (IV. 1.) an angle equal to the half of ADB; and about the vertex O, repeat the equal triangles AOB, AOE, EOF, FOG, GOH, HOI, IOK, and join KB, to compose the octagon.

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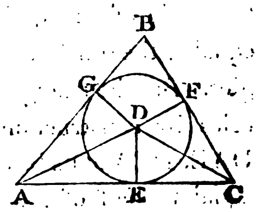


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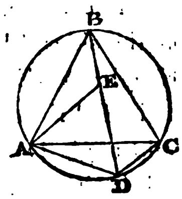
Let ABC be an equilateral triangle inscribed in a circle, and BD, AD, and CD chords drawn from its three corners to a point D in the circumference; BD, which crosses the base AC, is equal to AD and CD taken together.

For, from BD cut off DE equal to DA, and join AE.




The angle ADB is equal to ACB (III. 9.) in the same segment, which, being the angle of an equilateral triangle, is equal to the third part of two right angles. But the triangle ADE being isosceles by construction, has the angles DAE, DEA at its base equal, and each of them, therefore, equal to half of the remaining two-thirds of two right angles, or equal to one-third part. Consequently ADE is equilateral, and its angle DAE equal to CAB; take CAE from both, and there remains the angle DAC equal to EAB; but the angle ABD is equal to ACD in the same segment. And thus the triangles ADC and AEB have the angles DAC, DCA equal to EAB, EBA, and the interjacent side AC equal to AB; they are consequently equal, and the side DC is equal to EB. But DE was made equal to DA; wherefore DA and DC are together equal to DE and EB, or to DB.

## PROP. IX. PROB.

In and about a given circle, to inscribe and circumscribe a square.

Let EADB be a circle in which it is required to inscribe a square.

Draw the diameter AB, cross it (I. 4. cor.) by the perpendicular ED through the centre, and join AD, DB, BE, and EA: The inscribed figure ADBE is a square.

The angles about the centre C, being right angles, are equal to each other, and are, therefore, subtended by equal

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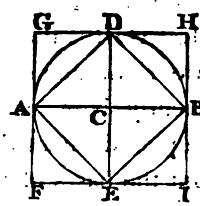



Next, let it be required to circumscribe a square about the circle.

Apply tangents FG, GH, HI, and FI at the extremities of the perpendicular diameters: These will form a square.

For all the angles of the quadrilateral figure CG, being together equal to four right angles, and those at C, A, and D being each a right angle, the remaining angle at G is also a right angle, and CG is a rectangle; but it is likewise a square, since AC is equal to CD. In the same manner, CH, CI, and CF are proved to be squares; the sides FG, GH, HI, and IF of the exterior figure, being therefore the doubles of equal lines, are mutually equal, and the angle at G being a right angle, FH is consequently a square.

*Cor.* Hence the circumscribing square is double of the inscribed square, and this again is double of the square constructed on the radius of the circle.

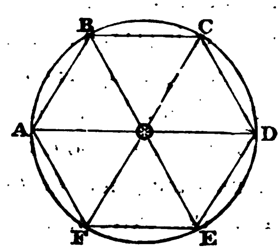
## PROP. X. PROB.

To inscribe a regular hexagon in a given circle.

Let it be required, in the circle FBD, to inscribe a hexagon.

Draw the radius OA, on which construct the equilateral triangle ABO, and repeat the equal triangles about the vertex O: These triangles will compose a hexagon.

For the triangle ABO, being equilateral, each of its angles AOB is the third part of two right angles, and consequently six of such angles may be placed about the centre O. But the bases of the triangles AOB, BOC, COD, DOE, EOF, and FOA form the sides of the figure, and the angles at those bases its internal angles; wherefore it is a regular hexagon.

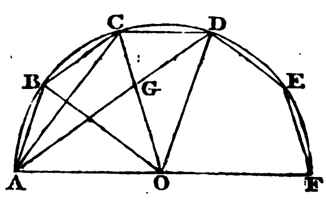



## PROP. XI. THEOR.

The square of the side of a pentagon inscribed in a circle, is equivalent to the squares of the sides of the inscribed hexagon and decagon.

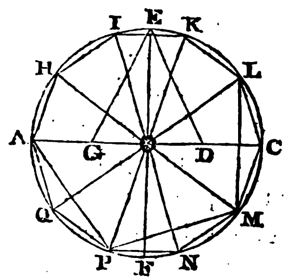
Let ABCDEF be half of a decagon inscribed in a circle whose diameter is AF; the square of AC, the side of an inscribed pentagon, is equivalent to the square of AB the side of the inscribed decagon, and of the square of the radius AO, or the side of an inscribed hexagon.

For join AD, and draw OB, OC, and OD. Since the arc DEF is double of AB, the angle AOB at the centre is evidently equal to OAD or OAG at the circumference; and because the arc BCDEF again is double of DEF, the angle OAB at the circumference is likewise equal to AOG at the centre. Whence the triangles AOB and AGO, having the angles OAB and AOB equal to AOG and OAG, and the interjacent side AO common, are equal, and therefore



the base AB is equal to OG. Consequently, GAO is an isosceles triangle, having each of the angles at its base double the vertical angle; wherefore (IV. 3.) OG is equal to the greater segment of the side AO divided by a medial section. But (II. 18. cor.) the square of AC exceeds the square of AO by the difference of the squares of the segments of the base BC, made by a perpendicular from the vertex A, or the rectangle under the sum and difference of those segments. Now this perpendicular bisecting the base, the difference of the segments is evidently CG. Wherefore the square of AC, the side of the inscribed pentagon, is equivalent to the square of AG or AO, the side of the inscribed hexagon, together with the rectangle OC, CG, or the square of OG or AB, the side of the inscribed decagon.

*Cor.* Hence the sides of the inscribed decagon and pentagon may be found by a single construction. For draw the perpendicular diameters AC and EF, bisect OC in D, join DE, make DG equal to it, and join GE. It is evident that AO is cut medially in G, and consequently that OG is equal to a side of the inscribed decagon. But GOE being a right-angled triangle,



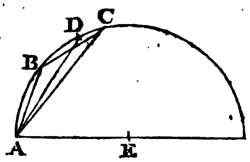

the square of GE is equivalent to the squares of GO and OE, or the squares of the sides of the decagon and hexagon; whence GE is equal to the side of the inscribed pentagon.

It is also evident, that the line CG compounded of the sides of the hexagon and of the decagon, and corresponding to AGD in the preceding figure, is equal to the chord of the triple arc, or of three-tenths of the whole circumference.

## PROP. XII. PROB.

**In a given circle, to inscribe regular polygons of fifteen and of thirty sides.**

Let AB and BC be the sides of an inscribed decagon, and AD the side of a hexagon likewise inscribed; the arc BD will be the fifteenth part of the circumference of the circle, and DC the thirtieth part.




For, if the circumference consisted of thirty equal portions, the arc AB would be equal to three of these, and the arc AD to five; consequently the excess BD is equal to two of these portions, or it is the fifteenth part of the whole circumference. Again, the double arc ABC being equal to six portions, and ABD to five, the defect DC is equal to one portion, or to the thirtieth part of the circumference.

*Cor.* From the inscription of the square, the pentagon, and the hexagon,—may be derived that of a variety of other regular polygons: For, by continually bisecting the intercepted arcs and inserting new chords, the inscribed figure will, at each successive operation, have the number of its sides doubled. Hence polygons will arise of 6, 8, and 10 sides; then of 12, 16, and 20; next of 24, 32, and 40; again, of 48, 64, and 80; and so forth repeatedly. The excess of the arc of the hexagon above that of the decagon, gives the arc of a fifteen-sided figure; and the continued bisection of this arc will mark out polygons with 30, 60, or 120 equal sides, in perpetual succession.

# RUDIMENTS

OF

# PLANE GEOMETRY.

## BOOK V.

### DEFINITIONS.

1. Straight lines drawn from or towards the same point, are termed *diverging* or *converging* lines.

2. Straight lines are divided *similarly*, when their corresponding segments have the same ratio.

3. A straight line is said to be cut in *extreme and mean ratio*, when the one segment is the mean proportional between the other segment and the whole line.

4. The *area* of a figure is the quantity of space which its surface occupies.

5. *Similar* figures are such as have their angles respectively equal, and the containing sides proportional.

6. If two sides of a rectilineal figure be the extremes of an analogy, of which the means are two corresponding sides in another rectilineal figure; those figures are said to have their sides *reciprocally* proportional.

## PROP. I. THEOR.

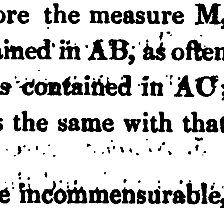
### Parallels cut diverging lines proportionally.

The parallels DE and BC cut the diverging lines AB and AC into proportional segments.

Those parallels may lie on the same side of the vertex, or on opposite sides; and they may consist of two, or of more straight lines.

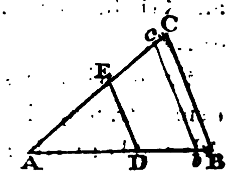
Let the two parallels DE and BC intersect the diverging lines AB and AC; then are AB and AC cut proportionally, in the points D and E, or AD : AB : : AE : AC.

For if AD be commensurable with AB, find (Introd. 14.) their common measure M, which repeat from the vertex A to B, and, from the corresponding points of section in AD and AB, draw the parallels FI, GK, and HL. It is evident that these parallels must also divide the straight lines AE and AC equally; for, if from the several points of section I, K, &c. lines were drawn parallel to the base, a series of triangles (1. 14. cor.) would be found all equal to AFI. Wherefore the measure M, or AF the submultiple of AD, is contained in AB, as often as AI, the like submultiple of AE, is contained in AC; consequently the ratio of AD to AB is the same with that of AE to AC.




But if the segments AD and AB be incommensurable, they may still be expressed numerically, to any degree of precision. For AD being divided into equal sections, these parts, continued towards B, will, together with some residual portion, compose the whole of AB. Let this division of AD extend through DB as far as *b*, and draw

the parallel $bc$. Let the parts of AD and AB be again subdivided, and the corresponding remainder $bB$ will evidently be diminished; consequently, at each successive subdivision, the terminating parallel $bc$ will approximate continually to BC. Wherefore, by repeating this process of exhaustion, the divided lines $Ab$ and $Ac$ will approach their limits AB and AC, nearer than any finite or assignable interval. Consequently, from the preceding demonstration, AD : AB : : AE : AC.

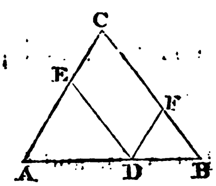



And since AD : AB : : AE : AC, it follows by conversion, that AD : DB : : AE : EC, and again, by composition, that AB : DB : : AC : EC.

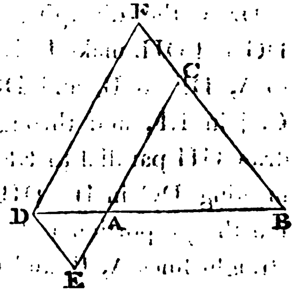
## PROP. II. THEOR.

Diverging lines are proportional to the corresponding segments into which they divide parallels.

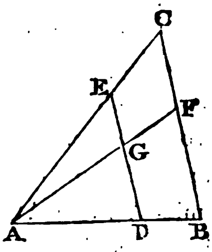
Let two diverging lines AB and AC cut the parallels BC and DE; then AB : AD : : BC : DE.




For draw DF parallel to AC. And, by the last Proposition, the parallels AC and DF must cut the straight lines AB and BC proportionally, or AB : AD : : BC : CF. But CF is equal (I. 26.) to the opposite side DE of the parallelogram DECF; and consequently AB : AD : : BC : DE.



Next, let more than two diverging lines, AB, AF and AC intersect the parallels BC and DE; the segments BF and FC have respectively to DG and GE the same ratio as AB has to AD.



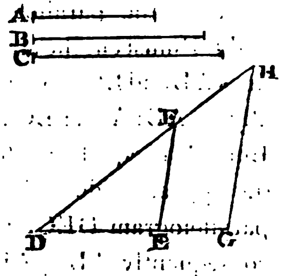

From what has been already demonstrated, it appears, that AB : AD : : BF : DG, and also that AF : AG : : FC : GE. But by the last Proposition, AB : AD : : AF : AG; wherefore AB : AD : : FC : GE. The same mode of reasoning, it is obvious, might be extended to any number of sections. Whence AB : AD : : BF : DG : : FC : GE.

*Cor*. Hence by parallels a given straight line is cut into segments proportional to those of a divided line.

## PROP. III. PROB.

To find a fourth proportional to three given straight lines.

Let A, B, and C be three straight lines, to which it is required to find a fourth proportional.




Draw the diverging lines DG and DH, make DE equal to A, DF to B, and DG to C, join EF, and through G draw GH parallel to EF and meeting DG in H; DH is a fourth proportional to the straight lines A, B, and C.

For the diverging lines DG and DH are cut proportionally by the parallels EF and GH (VI. 1.), or DE : DF : : DG : DH, that is, A : B : : C : DH.

*Cor.* If the mean terms B and C be equal, it is obvious that DG will become equal to DF; and that DH will be found a third proportional to the two given terms A and B.

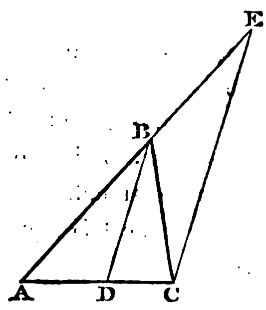
## PROP. IV. THEOR.

**A straight line which bisects, either internally or externally, the vertical angle of a triangle, will divide its base into segments, that are proportional to the adjacent sides of the triangle.**

Let the straight line BD bisect the vertical angle of the triangle ABC; it will cut the base AC into segments which have the same ratio as the adjacent sides, or AD : DC : : AB : BC.

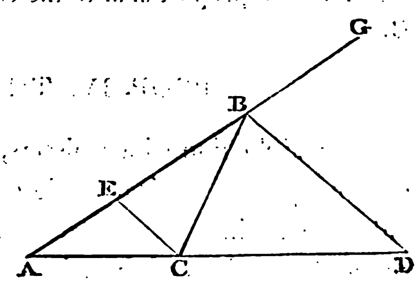
For through C draw CE parallel to DB, and meeting the production of AB in E.

Because DB and CE are parallel, the exterior angle ABD is equal to BEC, and the alternate angle DBC equal to BCE (I. 22.); wherefore the angle ABD being equal by hypothesis to DBC, the angle BEC is equal to BCE, and consequently the triangle CBE is isosceles, or BE is equal to BC. But the parallels DB and CE cut the diverging lines AC and AE proportionally (VI. 1.), or AD : DC : : AB : BE; that is, since BE=BC, AD : DC : : AB : BC



Again, let the vertical line BD bisect the exterior angle CBG of the triangle; it will divide the base AC into external segments AD and DC, which are also proportional to the adjacent sides AB and BC.

For through C draw CE parallel to DB, and meeting AB in E.



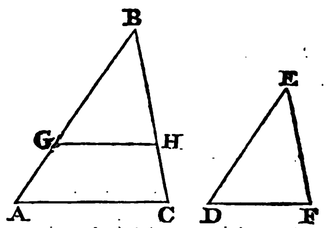

The equal angles GBD and DBC are, from the properties of parallel straight lines, respectively equal to BEC and BCE, and consequently the triangle CBE is isosceles, or the side BC is equal to BE. And since the diverging lines AD and AB are cut by the parallels DB and CE proportionally, AD : DC : : AB : BE or BC.

## PROP. V. THEOR.

**Triangles are similar, which have their corresponding angles equal.**

Let the triangles ABC and DEF have the angle CAB equal to FDE, CBA to FED, and consequently (I. 30.) the remaining angle BCA equal to EFD; these triangles are similar, or the sides in both that contain equal angles are proportional.

For make BG equal to ED, and draw GH parallel to AC.




Because GH is parallel to AC, the exterior angle BGH is equal (I. 9.) to BAC, or to EDF; and the angle at B is, by hypothesis, equal to that at E, and the interjacent side BG was made equal to ED; wherefore (I. 14. cor.) the triangle GBH is equal to DEF. But the diverging lines BA and BC being cut proportionally by the parallels AC and GH (V. 1.), AB is to BC as BG to BH, or as ED to EF. Again, those diverging lines being proportional to the intercepted segments AC and GH of the parallels (V. 2.), AB is to BG as AC is to GH, and alternately AB is to AC as BG is to GH, or as ED to DF. In the same manner, as BC is to BH so is AC to GH, and alternately, as BC is to AC so is BH or EF to GH or DF. And thus, the sides opposite to equal angles in the triangles ABC and DEF are the homologous terms of a proportion.

## PROP. VI. THEOR.

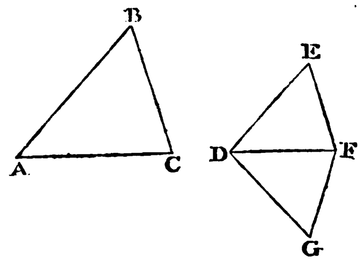
Triangles which have the sides about two of their angles proportional, are similar.

In the triangles ABC and DEF, let AB : AC : : DE : DF and BC : AC : : EF : DF; then is the angle BAC equal to EDF, and the angle BCA equal to EFD.

For (I. 4.) draw DG and FG, making angles FDG and DFG equal to CAB and ACB.

By the last Proposition, the triangle ABC is similar to

DGF, and consequently AB : AC : : DG : DF; but by hypothesis, AB : AC : : DE : DF, and hence, from identity of ratios, DG : DF : : DE : DF, or DG is equal to DE. In the same manner, BC : AC : : EF : DF, and BC : AC : : GF : DF; whence EF : DF : : GF : DF, and EF is equal to FG.




Wherefore the triangles DEF and DGF, having thus the sides DE and EF equal to DG and FG, and the side DF common to both, are (I. 17.) equal; consequently the angle EDF is equal to FDG or BAC, and the angle EFD is equal to DFG or BCA.

## PROP. VII. THEOR.

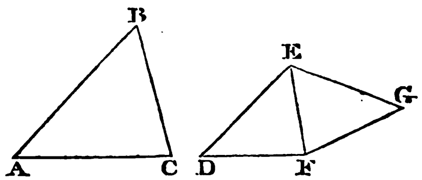
Triangles are similar, if each have an equal angle, and its containing sides proportional.

In the triangles BAC and EDF, let the angle ABC be equal to DEF, and the sides which contain the one be proportional to those which contain the other, or AB : BC : : DE : EF; the triangles BAC and EDF are similar.

For, from the points E and F, draw EG and FG, making the angles FEG and EFG equal to CBA and BCA.

The triangles BAC and EGF, having thus their corresponding angles equal, are similar (V. 5.), and therefore AB : BC : : EG : EF. But by hypothesis, AB : BC : :

ED : EF; wherefore EG : EF : : ED : EF, and consequently EG is equal to ED. Hence the triangles GFE




and DFE, having the side EG equal to ED, EF common to both, and the contained angle GEF equal to ABC or DEF, are equal (I. 14. cor.), and therefore the angle EFG or BCA is equal to EFD; consequently the remaining angles BAC and EDF of the triangles ABC and DEF are equal (I. 11.), and these triangles are (V. 5.) similar.

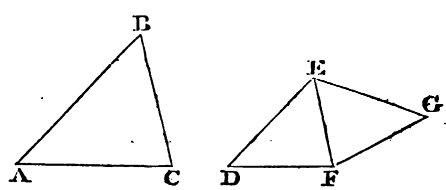
## PROP. VIII. THEOR.

Triangles are similar, which have each an equal angle, and the sides opposite to that and to another angle of the same character, proportional.

Let the triangles CAB and FDE have the angle ABC equal to DEF, and the sides opposite to these and to the angles at A and D proportional, or BC : AC : : EF : FD; while those angles are both of them acute or obtuse, the triangles ABC and DEF are similar.

For, from the points E and F draw EG and FG, making the angles FEG and EFG equal to ABC and BCA.

The triangle ABC is evidently similar to GEF,




and BC : CA : : EF : FG; but, by hypothesis, BC : CA : :

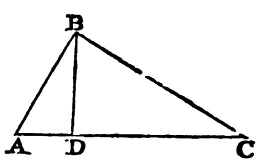
EF : FD, and therefore EF : FG : : EF : FD, and FG is equal to FD. Whence the triangles EGF and EDF, having the angle FEG equal to FED, the side FG equal to FD, EF common, and the angles A and D of the same character, are equal (I. 17.); consequently the angle GFE or ACB is equal to DFE, and therefore (V. 5.) the triangles ABC and DEF are similar.

## PROP. IX. THEOR.

A perpendicular let fall upon the hypotenuse of a right-angled triangle from the opposite vertex, will divide it into two triangles that are similar to the whole and to each other.

Let the triangle ABC be right angled at B, from which the perpendicular BD falls upon the hypotenuse AC; the triangles ABD and DBC, thus formed, are similar to each other, and to the whole triangle ACB.

For the triangles ADB and ACB, having the angle BAC common, and the right angle ADB equal to ABC, are similar (V. 5.) Again, the triangles DBC and ACB are similar, since they have the angle BCD common, and the right angle BDC equal to ABC. The triangles ABD and DBC being, therefore, both similar to the same triangle ABC, are evidently similar to each other.




*Cor.* Hence any side AB of a right-angled triangle is a mean proportional between the hypotenuse AC and the

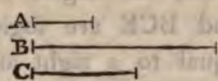
adjacent segment AD, formed by a perpendicular let fall upon it from the opposite vertex; and the perpendicular BD itself is a mean proportional between those segments AD and DC of the hypotenuse. For the triangles ABC and ADB being similar, AC : AB : : AB : AD; and the triangles ABC and BDC being similar, AC : BC : : BC : CD; again, the triangles ADB and BDC are similar, and therefore AD : DB : : DB : DC.

### PROP. X. PROB.

To find the mean proportional between two given straight lines.

Let it be required to find the mean proportional between the straight lines A and B.

Find C (III. 16. cor. 1.) the side of a square which is equivalent to the rectangle contained by A and B; C is the mean proportional required.



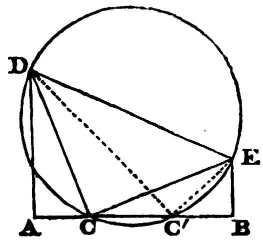

For since $C^2 = AB$, it follows (Introd. Prop. 3.) that A : C : : C : B.

### PROP. XI. PROB.

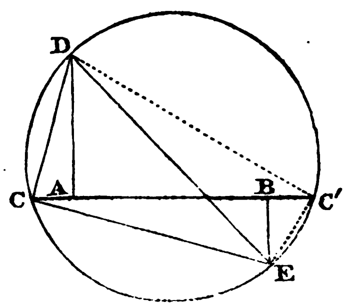
To divide a straight line, whether internally or externally, so that the rectangle under its segments shall be equivalent to a given rectangle.

Let AB be the straight line which it is required to cut, so that the rectangle under its segments shall be equivalent to a given rectangle.

From the extremities of AB, erect the perpendiculars AD and BE, equal to the sides of the given rectangle, and in the same or in opposite directions, according as the line is to be cut internally or internally; join DE, on which, as a diameter, describe a circle meeting AB or its extension in the point C: AC and CB are the segments required.




For join DC and CE. The angle DCE, being contained in a semicircle, is a right angle (III. 11.), and therefore, in both cases, the angles ACD and BCE are together equal to a right angle. But the angles ACD and CDA are likewise together equal to a right angle (I. 30. cor. 1.); and consequently the angles BCE and CDA are equal. Wherefore the right-angled triangles CBE and CAD, having the acute angle ADC equal to BCE, are similar (V. 5.); whence AC : AD : : BE : CB, and (Introd. Prop. 3.) AC.CB = AD.BE.

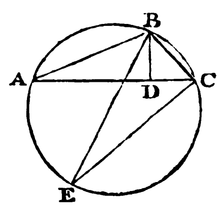



*Cor.* In the case of internal section, there is evidently a limitation; for when the circle merely touches AB, the points C and C′ coincide in the middle, and the rectangle under AD and BE becomes equivalent to the square of the half of that line. If the rectangle were to exceed that square, the circle described on the diameter DE would not meet AB.

## PROP. XII. THEOR.

The rectangle under any two sides of a triangle is equivalent to the rectangle under the perpendicular let fall on the base and the diameter of the circumscribing circle.

Let ABC be a triangle, about which is described a circle having the diameter BE; the rectangle under the sides AB and BC is equivalent to the rectangle under BE and the perpendicular BD let fall from the vertex of the triangle upon the base AC.




For join CE. The angle BAD is equal to BEC (III. 9.), since they both stand upon the same arc BC, and the angle ADB, being a right angle, is (III. 11.) equal to ECB, which is contained in a semicircle. Wherefore the triangles ABD and EBC, being thus similar (V. 5.), AB : BD : : EB : BC, and consequently (Introd. Prop. 3.) AB.BC=EB.BD.

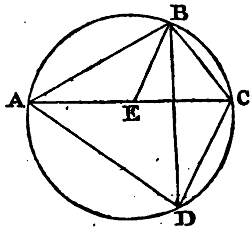
## PROP. XIII. THEOR.

The rectangles under the opposite sides of a quadrilateral figure inscribed in a circle, are together equivalent to the rectangle under its diagonals.

In the circle ABCD, let a quadrilateral figure be inscribed, and join the diagonals AC, BD; the rectangles AB, CD and BC, AD, are together equivalent to the rectangle AC, BD.

For (I. 4.) draw BE, making an angle ABE equal to CBD.

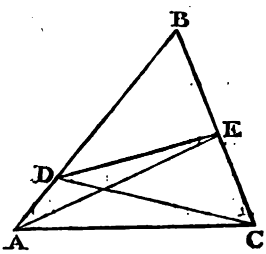
The triangles AEB and DCB, having thus the angle ABE equal to DBC, and the angle BAE or BAC equal (III. 9.) to BDC in the same segment of the circle, are similar (V. 5.), and hence AB : AE : : BD : CD; whence (Intr.) AB.CD = AE.BD. Again, because the angle ABE is equal to DBC, add EBD to each, and the whole angle ABD is equal to EBC; and the angle ADB is equal to ECB in the same segment; wherefore the triangles DAB and CEB are similar (V. 5.), and AD : BD : : EC : BC, and consequently BC.AD = EC.BD. Whence collectively the rectangles AB, CD and BC, AD are together equal to the rectangles AE, BD and EC, BD, that is, to the whole rectangle AC, BD.




## PROP. XIV. THEOR.

**Triangles which have a common angle, are to each other in the compound ratio of the containing sides.**

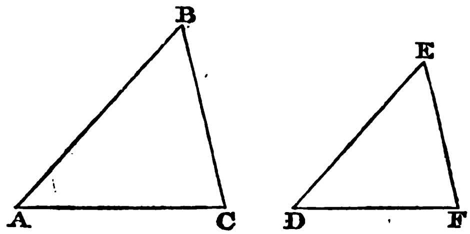
Let ABC and DBE be two triangles, having the same or an equal angle at B; ABC is to DBE in the ratio compounded of that of BA to BD, and of BC to BE.




For join AE and CD. The ratio of the triangle ABC to DBE may be conceived as compounded of that of ABC to DBC, and of DBC to DBE. But the triangle ABC is

to DBC, as the base BA to BD; and, for the same reason, the triangle DBC is to DBE, as the base BC to BE; consequently the triangle ABC is to DBE in the ratio compounded of that of BA to BD, and of BC to BE, or (Introd. Prop. 12.) in the ratio of the rectangle under BA and BC to the rectangle under BD and BE.

*Cor*. Hence similar triangles are in the duplicate ratio of their homologous sides. For, if the angle at B be equal to that at E, the triangle ABC is to DEF in the ratio




compounded of that of AB to DE, and of CB to FE; but, these triangles being similar, the ratio of AB to DE is the same as that of CB to FE (V. 5.), and consequently the triangle ABC is to DEF in the duplicate ratio of AB to DE, or (V. 13.) as the square of AB to the square of DE.

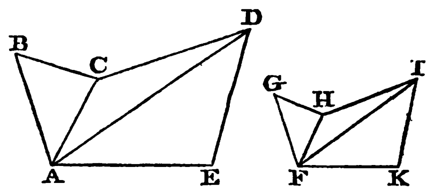
## PROP. XV. PROB.

On a given straight line, to construct a rectilineal figure similar to a given rectilineal figure.

Let FK be a straight line, on which it is required to construct a rectilineal figure similar to the figure ABCDE.

Join AC and AD, dividing the given rectilineal figure into its component triangles. From the points F and K draw FI and KI, making the angles KFI and FKI equal to EAD and AED; from F and I draw FH and IH making the angles IFH and FIH equal to DAC and ADC; and, lastly, from F and H draw FG and HG making the angles HFG and FHG equal to CAB and ACB. The figure FGHIK is similar to ABCDE.

For the several triangles KFI, IFH, and HFG, which compose the figure FGHIK, are, by the construction, evidently similar to the triangles EAD, DAC, and CAB, in-




to which the figure ABCDE was resolved. Whence EK : KI : : AE : ED; also KI : IF : : ED : DA, and IF : IH :: DA : DC, and consequently (Introd. Prop. 7.) KI : IH : : ED : DC. Again, IH : HF :: DC : CA, and HF : HG : : CA : CB; and hence IH : HG :: DC : CB. But HG : GF : : CB : BA; and the ratio of GF to FK, being compounded of that of GF to FH, of FH to FI, and of FI to FK, is the same with the ratio of BA to AE, which is compounded of the like ratios of BA to AC, of AC to AD, and AD to AE. Wherefore all the sides about the figure FGHIK are proportional to those about ABCDE; but the several angles of the former, having a like composition, are respectively equal to those of the latter. Whence the figure FGHIK is similar to the given figure.

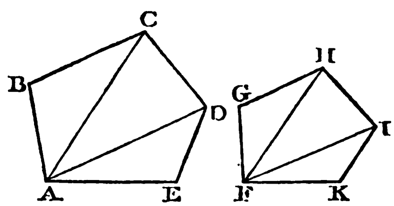
*Cor.* Hence similar rectilineal figures may be divided into corresponding similar triangles.

## PROP. XVI. THEOR.

**Of similar figures, the perimeters are proportional to the corresponding sides, and the areas, to their squares.**

Let ABCDE and FGHIK be similar polygons, which have the corresponding sides AB and FG; the perimeter, or linear boundary, ABCDE is to the perimeter FGHIK, as AB to FG, BC to GH, CD to HI, DE to IK, or EA to KF; but the area of ABCDE, or the contained surface, is to the area of FGHIK, in the duplicate ratio of those homologous terms, or of AB to FG, of BC to GH, of CD to HI, of DE to IK, or of EA to KF.

For, by drawing the diagonals AC, AD in the one, and FH, FI in the other, these polygons will be resolved into similar triangles. Whence the several analogies AB: BC:: FG: GH, BC: AC:: GH: FH,




AC : CD : : FH : HI, CD : AD : : HI : FI, and AD : DE : : FI : IK; wherefore, by equality and alternation, AB : FG : : BC : GH : : CD : HI : : DE : IK : : AE : FK, and consequently (Introd. Prop. 8.) as one of the antecedents AB, BC, CD, DE or AE, is to its consequent FG, GH, HI, IK or FK, so is the amount of all those antecedents, or the perimeter ABCDE, to the amount of all the consequents, or the perimeter FGHIK.

Again, the triangle CAB is to the rectangle HFG in

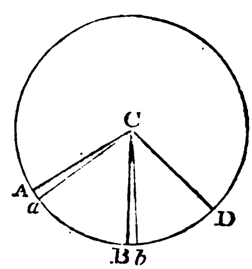
the duplicate ratio of AB to FG, the triangle DAC is to the triangle IFH in the duplicate ratio of AC to FH, or of AB to FG, and the triangle EAD is to KFI in the duplicate ratio of AD to FI or of AB to FG; wherefore the aggregate of the triangles CAB, DAC, and EAD, or the area of the polygon ABCDE, is to the aggregate of the triangles HFG, IFH, and KFI, or the area of the polygon FGHIK, in the duplicate ratio of AB to FG, of BC to GH, of CD to HI, or of DE to IK.

## PROP. XVII. THEOR.

The arcs of a circle are proportional to the angles which they subtend at the centre.

Let the radii CA, CB, and CD intercept arcs AB and BD; the arc AB is to BD, as the angle ACB to BCD.

For (I. 5.) bisect the angle ACB, bisect again each of its halves, and continue the operation indefinitely. An angle AC*a* will be thus obtained which is less than any assignable angle. Let this angle AC*a* or BC*b* (I. 4.) be repeatedly applied about the point C, from BC towards DC; it must hence, by its multiplication, fill up the angle BCD, nearer than any possible difference. But the elementary angle AC*a* being equal to BC*b*, the corresponding arc A*a* is equal to B*b*. Consequently this arc A*a* and its angle AC*a*, are like measures of the arc AB and the angle ACB, and they are both contained equally in the arc BD and its corresponding angle BCD. Wherefore AB : BD : : ACB : BCD.

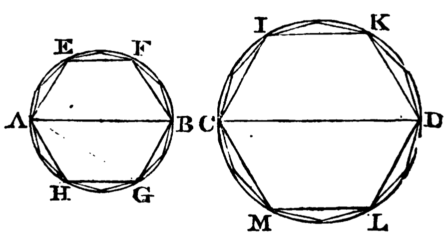


## PROP. XVIII. THEOR.

The circumference of a circle is proportional to the diameter, and its area to the square of that diameter.

Let AB and CD be the diameters of two circles; the circumference AFG is to the circumference CKL, as AB to CD; and the area contained by AFG is to the area contained by CKL, as the square of AB to the square of CD.

For inscribe the regular hexagons AEFBGH and CIKDLM. Because these polygons are equilateral and equiangular, they are similar; and consequently the diagonal AB is to the corresponding diagonal CD, as the perimeter AEFBGH to the perimeter CIKDLM. But this proportion must subsist, whatever be the number of chords inscribed in either circumference. Insert a dodecagon in each circle between the hexagon and the circumference, and its perimeter will evidently ap-

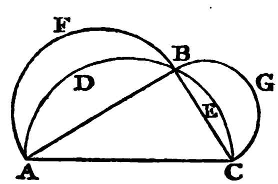



proach nearer to the length of that circumference. Proceeding thus, by repeated duplications,—the perimeters of the series of polygons that arise in succession, will conti-

nually approximate to the curvilineal boundary, which forms their ultimate limit. Wherefore this extreme term, or the circumference AEFBGH, is to the circumference CIKDLM, as the diameter AB to the diameter CD.

Again, the hexagon AEFBGH is to the hexagon CIKDLM in the duplicate ratio of the diagonal AB to the corresponding diagonal CD, or as the square of AB to the square of CD. Wherefore the successive polygons which arise from a repeated bisection of the intermediate arcs, and which approach continually to the areas of their containing circles, must have still that same ratio. Consequently the limiting space, or the circle AEFBGH, is to the circle CIKDLM, as the square of AB to the square of CD.

*Cor.* It hence follows, that if semicircles be described on the sides AB, BC of a right-angled triangle, and on the hypotenuse AC another semicircle be described, passing (III. 11.) through the vertex B, the crescents AFBD and BGCE are together equivalent to the triangle ABC. For, by the Proposition, the square of AC is to the square of AB, as the circle on AC to the circle on AB, or as the semicircle ADBEC to the semicircle AFB; and, for the same reason, the square of AC is to the square of BC, as the semicircle ADBEC to the semicircle BGC. Whence the square of AC is to the squares of AB and BC, as the semicircle ADBEC to the semicircles AFB and BGC. But




the square of AC is equivalent to the squares of AB and BC, and therefore the semicircle ADBEC is equivalent to the two semicircles AFB and BGC; take away the common segments ADB and BEC, and there remains the

triangle ABC equivalent to the two crescents or *lunes* AFBD and BGCE.

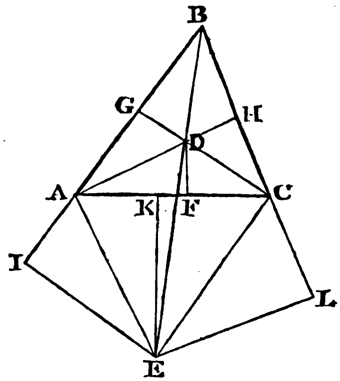
## PROP. XIX. THEOR.

The area of any triangle is a mean proportional between the rectangle under the semiperimeter and its excess above the base, and the rectangle under the separate excesses of that semiperimeter above the two remaining sides.

The area of the triangle ABC is a mean proportional between the rectangle under half the sum of all the sides and its excess above AC, and the rectangle under the excess of that semiperimeter above AB and its excess above BC.

For produce the sides BA and BC, draw the straight lines BE, AD, and AE bisecting the angles CBA, BAC, and CAI, join CD and CE, and let fall the perpendiculars DF, DG, and DH within the triangle, and the perpendiculars EI, EK, and EL without it.

The triangles ADF and ADG, having by construction the angle DAF equal to DAG, the angles F and G right angles, and the common side AD, are (I. 17. cor.) equal; for the same reason, the triangles BDG and BDH are equal. In like manner, it is proved, that the triangles AEI and AEK are equal, and also the triangles BEI and BEL. Whence the triangles CDH and CDF, having the side DH equal to DF, the side DC common, and the right angle CHD equal to CFD, are (I. 17. cor.) equal; and, for the same reason, the triangles CEK and CEL are equal. Wherefore the segments AF, FC and BG




are respectively equal to AC, CH and BH, and compose with them the whole sides of the triangle ABC. Consequently the segments AF, FC and BG, or AC and BG, is equal to the semiperimeter, and BG is thus its excess above the base. But the segments BH, HC and AG, or BC and AG, being likewise equal to the semiperimeter, AG is its excess above the side BC. Again, since the segments AK and CK of the base are equal to the productions AI and CL of the sides AB and BC, the equal lines BI and BL are together equal to the whole perimeter of the triangle, or each of them is equal to the semiperimeter, and AI is its excess above the side AB.

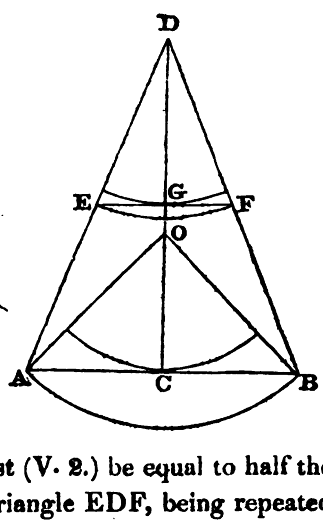
Now, because DG and EI, being perpendicular to BI, are parallel, BG : DG : : BI : EI (V. 2.), and consequently (Introd. Prop. 1.) BI combining with the two first terms of the analogy, and DG with the two last, BI × BG : BI × DG : : DG × BI : DG × EI. But since AD and AE bisect the angle BAC and its adjacent angle CAI, the angles GAD and EAI are together equal to a right angle, and equal, therefore, to IEA and EAI; whence the angle GAD is equal to IEA, and the right-angled triangles DGA and AIE are similar. Wherefore (V. 5.) DG : AG : : AI : EI and hence DG × EI = AG × AI; consequently BI × BG : DG × BI : : DG × BI : AG × AI. But the triangle ABC is composed of three triangles ADB, BDC, and CDA, which have the same altitude; and therefore its area is equal to the rectangle under the com-

mon perpendicular DG and half their bases AB, BC, and AC, or the semiperimeter BI. Whence the area of the triangle ABC is a mean proportional between the rectangle under BI and its excess above AC, and the rectangle under its excess above BC and that above AB.

## PROP. XX. PROB.

To convert a given regular polygon into another, which shall have the same perimeter, but double the number of sides.

It is evident that, by drawing lines from the centre of the inscribed or circumscribing circle to all the corners of a regular polygon, this may be divided into as many equal and isosceles triangles as it has sides. Let AOB be such a sector of the given polygon; from the centre O let fall the perpendicular OC, and produce it to D, till OD be equal to OA or OB, and join AD and BD. The isosceles triangle ADB is therefore (IV. 1.) constructed on the same base with AOB, but has only half the vertical angle. Consequently twice as many of such angles could be constituted about D, as were placed about O. Bisect AD and BD in E and F, and the straight line joining these points must (V. 2.) be equal to half the base AB. Wherefore the triangle EDF, being repeated about the vertex D, would form a regular polygon with twice as many sides as before, yet under the same extent



of perimeter, since each of those sides EF has only half the former length ACB.

*Cor.* 1. Hence DG, the radius of the circle which would inscribe the derived polygon, is half of CD, that is, half of the sum of OC and OA, the radii of the circles inscribing and circumscribing the given polygon. Again, since AOD is evidently isosceles, $AD^2 = 2OA.CD$ (II. 13. cor.), or $DE^2 = OA.\frac{1}{2}CD$, and consequently DE the radius of the circumscribing derived polygon is a mean proportional between OA and DG, the radius of the circle circumscribing the given polygon, and the radius of the circle inscribing the derived polygon.

*Cor.* 2. Hence the area of a circle is equivalent to the rectangle under its radius, and a straight line equal to half its circumference. For the surface of any regular circumscribing polygon, being composed of triangles such as EDF, which have all the same altitude DG, is equivalent (II. 1.) to the rectangle under DG, and half the sum of their bases, or the semiperimeter of the polygon. Therefore the circle itself, since it forms the ultimate limit of the polygon, must have its area equivalent to the rectangle under the radius or the limit of all the successive altitudes and the semicircumference, which limits also the corresponding semiperimeters.

From this proposition is derived a very simple and elegant method of approximating to the numerical expression for the area of a circle. Let the original polygon be a square, each side of which is denoted by unit; the component sector AOB is therefore a right-angled isosceles triangle, having the perpendicular OC, or the radius of the inscribed circle, equal to .5, and the radius OA of the circumscribing circle equal to $\sqrt{.5}$ or .7071067812. But DG, the radius of a circle inscribed in an octagon of

the same perimeter, is $= \frac{OA+OC}{2} = \frac{.5+.707106812}{2} =$ .6035533906; and DE the radius of the circle circumscribing that octagon, is $= \surd(OA.DG) = \surd(.603533906 \times .707106812) = .6532814824$. Again, the radius of the circle inscribed in a polygon of 16 sides with the same perimeter, is $= \frac{.6035533906+.6532814824}{2} = .62844174365$; and the radius of the circle circumscribing that polygon, is $= \surd(.62844174365 \times .6532814824) = .6407288619$. In like manner, the radii of circles inscribing and circumscribing the polygons of 32, 64, 128, &c. sides, under the same perimeter, are successively found, by an alternate series of arithmetical and geometrical means.

| No. of sides of the Polygon. | Radius of Inscribed Circle. | Radius of Circumscribing Circle. |
|---|---|---|
| 4 | .5000000000 | .7071067812 |
| 8 | .6035533906 | .6532814824 |
| 16 | .6284174365 | .6407288619 |
| 32 | .6345731492 | .6376435773 |
| 64 | .6361083633 | .6368755077 |
| 128 | .6364919355 | .6366836927 |
| 256 | .6365878141 | .6366357516 |
| 512 | .6366117828 | .6366237671 |
| 1024 | .6366177750 | .6366207710 |
| 2048 | .6366192730 | .6366200200 |
| 4096 | .6366196475 | .6366198348 |
| 8192 | .6366197411 | .6366197880 |
| 16384 | .6366197645 | .6366197763 |
| 32768 | .6366197704 | .6366197733 |
| 65536 | .6366197719 | .6366197726 |
| 131072 | .6366197722 | .6366197724 |
| 262144 | .6366197723 | .6366197724 |

But the final term may be discovered still more expeditiously; for, since the numbers in both columns are formed by taking successive means, those of the second column must each time be diminished by the fourth part of

the common difference, and consequently, from Introd. Prop. 10, the continued diminution will accumulate to one-third of that difference. Wherefore the ultimate radius of the inscribed and circumscribing circles is the third-part of the sum of a radius of inscription and of double the corresponding radius of circumscription. Thus, stopping at the polygon of 256 sides,

$$\frac{.63665878141+2(.6366357516)}{3} = .6366197724, \text{ the final}$$

result.

Hence the radius of a circle, whose circumference is 4, or the diameter of a circle whose circumference is 2, will be denoted by .6366197724; wherefore, reciprocally, the circumference of a circle whose diameter is 1, will be expressed by 3.1415926536, and its area, or that of the ultimate polygon, by .7853981434.

By the application of Conditional Equations, (Introd. Prop. 15.) the ratio of the diameter to the circumference is progressively denoted by 1 : 3, by 7 : 22, by 113 : 355, and by 1250 : 3927. The ratio of 1 to 3 is the rudest approximation, being the same as that of the diameter of the circle to the perimeter of its inscribed hexagon; the ratio of 7 to 22 is what was discovered by Archimedes; the ratio of 113 to 355, in which the three first odd digits appear in pairs, was originally proposed by Adrian Metius of Alkmaer, who died in 1636; and the ratio of 1250 to 3927, the same as 1 to 3.1416, which was known to the Arabians, and is now generally preferred in practice. Wherefore 3.1416, multiplied into the diameter of a circle, will denote its circumference, and .7854, multiplied into the square of the diameter, will give the numerical expression for its area, which is also to this square nearly as 11 to 14.

# GEOMETRICAL ANALYSIS.

Analysis is that procedure by which a proposition is traced, through a chain of necessary dependence, to some admitted principle, or some known operation. It is alike applicable to the investigation of truth contemplated in a *theorem*, or to the discovery of the construction required for a *problem*. Analysis, as its name indeed imports, is thus a sort of inverted form of solution. Setting out from the hypothesis advanced, it mounts, step by step, till it has reached a source already explored. The reverse of this process constitutes *Synthesis*, or *Composition*,—which is the mode usually employed for explaining the elements of science. Analysis, therefore, presents the medium of invention; while synthesis naturally directs the course of instruction.

## DEFINITIONS.

1. *Quantities* are said to be *given*, which are either exhibited, or may be found.

2. A *ratio* is said to be *given*, when it is the same as that of two given quantities.

3. *Points*, *lines*, and *spaces*, are said to be *given in position*, if they have always the same situation, and are either actually exhibited, or may be found.

4. A *circle* is *given in position*, when its centre is given: it is *given in magnitude*, if its radius be given.

5. *Rectilineal figures* are said to be *given in species*, when figures similar to them are given.

6. If a point vary its position according to some determined law, it will trace a line which is termed its *Locus*.

7. *Isoperimetrical* figures are such as have equal perimeters, or the same extent of linear boundary.

A variable quantity derived from another given or constant quantity, or which depends on it by some relation according to a given law, is necessarily confined between certain extreme limits. When it has acquired the greatest possible expansion, it is said to have reached a *maximum*; and when it has contracted into its lowest dimensions, it occupies the state of a *minimum*.

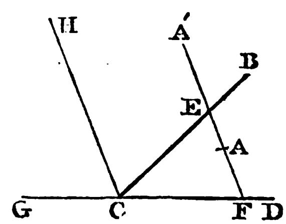
## PROP. I. PROB.

**Through a given point, to draw a straight line at equal angles with two inclined straight lines given in position.**

Let A be the given point, and CB, CD the inclined straight lines which are given in position.

### ANALYSIS.

Let those inclined lines meet in C; draw CH parallel to FE, and produce DC. The exterior angle GCH is equal to CFE, and ECH is equal to the alternate angle CEF, but the angle CFE is equal to CEF, and consequently GCH is equal to ECH, and the angle GCE is thus bisected by the straight line CH. Wherefore CH is given in position, and hence the parallel EF is also given.




### COMPOSITION.

Bisect the adjacent angle GCB by the straight line CH, and parallel to this draw EF through the given point A; the angle CEF is equal to CFE. For these angles are equal respectively to the exterior and to alternate angles GCH and ECH of the parallels CH and AF, and are consequently equal to each other.

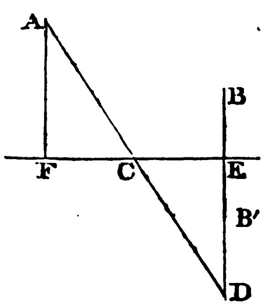
## PROP. II. PROB.

Through a given point, to draw a straight line, such that the segments intercepted by perpendiculars let fall upon it from two given points, shall be equal.

The points A, B, and C being given,—to draw a straight line FE through C, so that the parts CF and CE, cut off by the perpendiculars AF and BE, let fall from A and B, shall be equal.

### ANALYSIS.

Produce AC to meet the extension of BE in D. The right-angled triangles AFC and DEC, having the vertical angle ACF equal to DCE, and the side CF by hypothesis equal to CE, are equal, and hence the side CA is equal to CD. But CA is evidently given; wherefore CD and the point D are given; BD is consequently given, and hence the line CE at right angles to it is given.




### COMPOSITION.

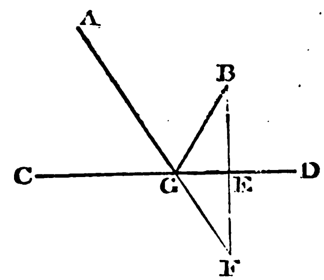
Join AC, and produce it till CD be equal to it; join BD, on which, from C, let fall the perpendicular FCE, which is the line required. For, draw AF parallel to BD, and the triangles FAC and EDC, having the angles ACF, AFC equal to DCE, DEC, and the side AC equal to CD, are equal, and consequently CF is equal to CE.

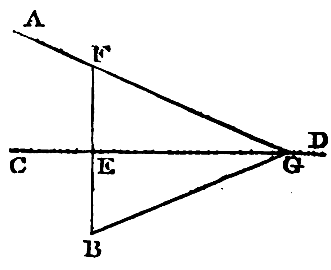
## PROP. III. PROB.

**From two given points, to draw straight lines, making equal angles at the same point in a straight line given in position.**

Let A, B be two given points, and CD a straight line given in position; it is required to draw AG and GB to some point in it, so that the angles AGC and BGD on either side shall be equal.

### ANALYSIS.




From B, one of the given points, let fall the perpendicular BE, and produce it to meet AG, or its extension in F. The angle BGE, being equal to AGC, is equal to the vertical angle FGE, the right angle BEG is equal to FEG, and the side GE is common to the triangles GBE and GFE, which are therefore equal, and hence the side BE is equal to FE. But the perpendicular BE is given, and consequently EF is given both in position and magnitude; whence the point F is given, and therefore G the intersection of the straight line AF and CD.

## COMPOSITION.

Let fall the perpendicular BE, and produce it equally on the opposite side, join AF meeting CD in G; AG and BG are the straight lines required.

For the triangles GBE and GFE, having the side BE equal to FE, GE common, and the contained angle BEG equal to FEG, are equal; and consequently the angle BGE is equal to FGE or AGC.

When the points A and B lie on the same side of the line CD, the problem will always admit of a solution, since AGF must cross CD. If those points lie on opposite sides of CD and equidistant from it, AFG will become parallel to CD, and consequently the problem will fail, unless they range in the same perpendicular, or the point F coincide with A, and the solution is indefinite, and every point in CD answers the conditions.

## PROP. IV. PROB.

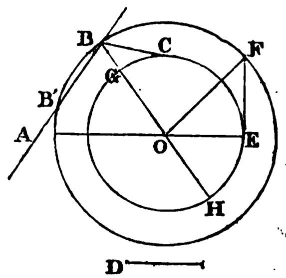
From a straight line given in position to draw, equal to a given line, a tangent to a given circle.

Let it be required in the straight line AB to find a point B, from which the tangent BC, drawn to the circle having O for its centre, shall be equal to D.

## ANALYSIS.

Through the centre O, draw BGOH, the square of BC is equivalent to the rectangle BG, BH, or to the excess

of the square of OB above that of OG, whence the square of OB is equivalent to the squares of OC and BC, and is therefore given. Consequently, OB itself is given, and the point B, being the intersection of AB with a given circle, is likewise given.




## COMPOSITION.

From O draw any radius OE, erect the perpendicular EF equal to D, join OF, and from O as a centre, describe through F a circle meeting AB in B, from which the tangent BC drawn to the interior circle is equal to D.

For $BC^2 = GB.BH = OB^2 - OG^2$, or $OB^2 = BC^2 + OG^2$; but $OF^2$ or $OB^2 = OE^2 + EF^2$, and $BC^2 + OG^2 = OE^2 + EF^2$, or $BC^2 = EF^2$, and consequently BC is equal to EF or D.

## PROP. V. PROB.

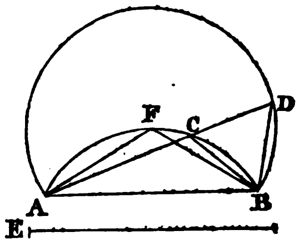
**In a given arc of a circle, to inflect two chords that shall be together equal to a given straight line.**

Let AFB be a given segment, it is required to inscribe the chords AC and BC, whose sum shall be equal to the line E.

## ANALYSIS.

Produce AC till CD be equal to BC, and join BD. The triangle BCD is evidently isosceles, and consequent-

ly the exterior angle ACB is double of CDB; but ACB is given, and therefore its half, or ADB, is given. Whence ADB is contained in a given segment, and the inflected line AD, being equal to E, is given in position; wherefore the point C and the chords AC and BC are given.




## COMPOSITION.

Bisect the arc AFB in the point F, from which centre, with the chord FA or FB as a radius, describe a circle, and inflect ACD equal to E; AC and BC being joined, are the chords required.

For join BD. The angle AFB at the centre is double of ADB at the circumference, and consequently ACB is likewise double of ADB or CDB; but ACB is equal to the two angles CDB and CBD, which are hence equal to the double of CDB, and therefore CDB and CBD are mutually equal, and contain an isosceles triangle. Whence BC is equal to CD, and AC and BC together are equal to AD or E.

It is evident that E, the sum of the inflected chords, can never exceed the diameter of the exterior circle, or the amount of the equal chords AF and BF. In other cases, the point D, and consequently C will have two positions.

This proposition exhibits both a *locus* and a *maximum*. If a chord AC be extended to D, such that the part CD shall be equal to the adjacent chord BC, the locus of the extremity D will be a circle, whose centre is the middle point F, having for its radii the equal chords AF and BF.

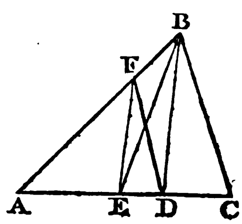
## PROP. VI. PROB.

**To bisect a given triangle, by a straight line drawn from a given point in one of its sides.**

Let it be required, from the point D, to draw DF, bisecting the triangle ABC.

### ANALYSIS.

If D be the middle of AC, the line DB drawn to the vertex will obviously divide the triangle into two equal portions. But, if not, bisect the side AC in E, and join EB, EF and BD. The triangle ABE is equal to EBC, and is consequently the half of ABC, wherefore ABE is equal to AFD, and, taking AFE from both, the remaining triangle EFB is equal to EFD; and since these triangles stand on the same base EF, they must have the same altitude, or EF is parallel to BD. But the points B and D being given, the straight line BD is given in position, and consequently the parallel EF is also given in position.




### COMPOSITION.

Having bisected AC in E and joined BD, draw EF parallel to it, meeting AB in F; the straight line DF divides the triangle ABC into two equal portions.

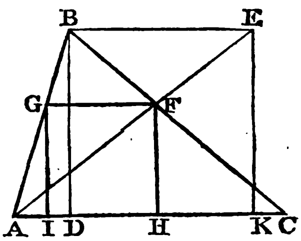
For join BE. Because BD is parallel to EF, the triangle EFB is equal to EFD; and, adding AFE to each, the triangle AFD is equal to ABE, that is, to the half of the triangle ABC.

## PROP. VII. PROB.

To inscribe a square in a given triangle.

Let ABC be the triangle in which it is required to inscribe a square IGFH.

### ANALYSIS.




Join AF, and produce it to meet a parallel to AC in E, and let fall the perpendiculars BD and EK.

Because EB is parallel to FG or AC, AF : AE : : FG : EB; and since the perpendicular EK is parallel to FH, AF : AE : : FH : EK. Wherefore FG : EB : : FH : EK; but FG=FH, and consequently EB=EK. Again, EK, being equal to BD, the altitude of the triangle ABC is given, and, therefore, EB is given both in position and magnitude; whence the point E is given, and the intersection of AE with BC is given, and consequently the parallel FG and the perpendicular FH are given, and thence the square IGFH.

### COMPOSITION.

From B draw BD perpendicular and BE parallel to AC, make BE equal to BD, join AE, intersecting BC in F, and complete the rectangle IGFH.

Because BE and EK are parallel to GF and FH, AE : AF : : BE : GF, and AE : AF : : EK : FH; wherefore BE : GF : : EK : FH; but BE=EK, and consequently GF=FH. It is hence evident that IGFH is a square.

## PROP. VIII. PROB.

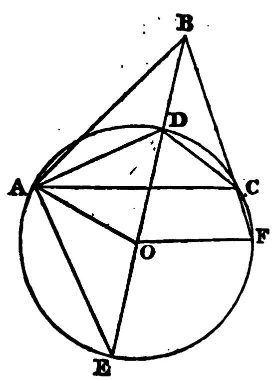
Given two sides of a triangle, and the distance of its vertex from the centre of the inscribed circle, to construct the triangle.

Let the sides AB and CB, together with the straight line BD drawn from the vertex B to D the centre of an inscribed circle be given, to construct the triangle ABC.

### ANALYSIS.

Join AD and DC, about the triangle ADC describe a circle, produce BD to E, and join AE.

Because D is the centre of the inscribed circle, the angle is bisected; but AEB or AED is equal to ACD, and ABD is equal to CDB, that is, to DCB. Wherefore the triangle BAE is similar to BDC, and BE : BA : : BC : BD; whence the rectangle under BE and BD, being equivalent to the rectangle under AB and BC, is given; but BD being given, BE and the point E are likewise given. Again, the angle DAE consists of DAC and CAE or CDE; of which, DAC is half the angle BAC, and CDE is equal to the angles CBD and DCB, or to half the angles ABC and ACB; consequently the angle DAE is half of all the angles of the triangle ABC, and is therefore a right angle, and stands on the diameter DE. The circle CDAE is hence given, with the points E, D and B,



and therefore BA and BC being given, the points A and C are likewise given, and thence the triangle ABC.

## COMPOSITION.

Produce BD to E, so that BD : BC : : BA : BE; on DE as a diameter describe a circle, and from B inflect to the circumference lines equal to BA and BC; then, AC being joined, ABC is the triangle required.

For produce BC to meet the circumference again in F, and join OA, OF. From the property of a circle, the rectangle BF, BC is equivalent to BD, BE; but, by construction, this rectangle BD, BE is equivalent to BA, BC, which is therefore equivalent to BF, BC, and consequently BA is equal to BF. Wherefore the triangle ABO has all its sides respectively equal to those of EBO, and these triangles are hence equiangular. The line BD consequently bisects the angle ABC; but since, by construction, BD : BC : : BA : BE, and the angle ABE is equal to CBD, the triangles BAE and BDC are similar; whence the angle AED, which is equal to ACD, is likewise equal to DCB, and CD bisects the angle ACB. The point D is, therefore, by its position, the centre of a circle inscribed within the triangle ABC.

## PROP. IX. PROB.

From the vertex of a given triangle, to draw a straight line which shall be a mean proportional between the segments of the base.

### CASE I.

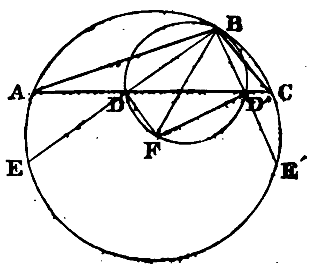
*When the section is internal.*

Let ABC be the given triangle, from the vertex of

which it is required to draw BD a mean proportional to the segments AD and DC of the base.

## ANALYSIS.

Describe a circle about the triangle, and produce BD to meet the circumference in E. Since BD is a mean proportional between AD and DC, its square is equivalent to the rectangle under AD and DC; but this rectangle is equivalent to the rectangle under BD and DE, which is therefore equivalent to the square of BD, and consequently BD itself is equivalent to DE. Wherefore FD being joined, is perpendicular to BE, and the right angle FDB is contained in a semicircle, whose intersection with the base of the triangle determines the position of the point D.




## COMPOSITION.

About the triangle ABC describe a circle, join its centre F with the vertex B, and on this straight line, as a diameter, describe another circle cutting the base of the triangle in the point D. Join BD, and it is the line required.

For produce BD to the circumference, and join FD.

The angle FDB is a right angle, and consequently BD is equal to DE. But the rectangle AD, DC is equivalent to BD, DE, or to the square of BD. Whence BD is a mean proportional between AD and DC, the segments of the base.

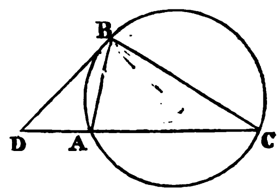
## CASE II.

*When the section is external.*

Let it be required, from the vertex B, to draw to a point beyond the base the straight line BD, a mean proportional to the segments DA and DC.

## ANALYSIS.

Describe a circle about the given triangle. And since, by hypothesis, the square of DB is equivalent to the rectangle under DA and DC, DB must touch the circumference. But this tangent BD is given in position, consequently is likewise its intersection D with the produced base.




## COMPOSITION.

The point D is now determined by a perpendicular from the extremity B of a diameter, the variable position of a right angle about the radius thus connecting both the cases.

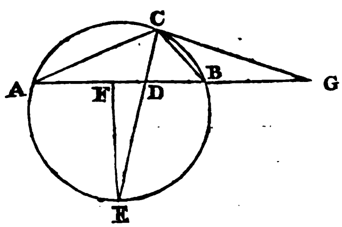
## PROP. X. PROB.

From two given points in the circumference of a given circle, to inflect, to another point in that circumference, straight lines which shall have a given ratio.

From the points A and B, let it be required to inflect AC and BC in a given ratio.

## ANALYSIS.

Draw CE bisecting the vertical angle ACB. Therefore AC : CB : : AD : DB, and consequently the ratio of




AD to DB is given, and thence the point D is given. But since the angle ACE is equal to BCE, the arc AE is equal to the arc EB, and therefore the point E is given. Whence the points E and D being given, the straight line EDC is given in position, and consequently the point C and the chords AC and BC are given.

## COMPOSITION.

Bisect the arc AEB in E, divide AB in the given ratio at D, join ED, and produce it to meet the opposite circumference in C; the chords AC and CB are in the given ratio.

For since the arc AE is equal to BE, the angle ACD is equal to BCD, and consequently AC : CB : : AD : DB, that is, in the given ratio.

It is evident, that the point E lies in the perpendicular which bisects AB; and FD being given, its extension DG is likewise given, such that FD.DG = AD.DB = CD.DE; whence FD : DE : : CD : DG, and the vertical angles FDE and CDG being equal, the triangles EFD and DCG are similar, and both right angled. The point of inflexion C of the proportional lines AC and BC in all the arcs described on AB will thus occur in the circumference of a semicircle, of which the diameter is DG, and DCG the included angle.

This conclusion anticipates the twentieth Proposition of this Book.

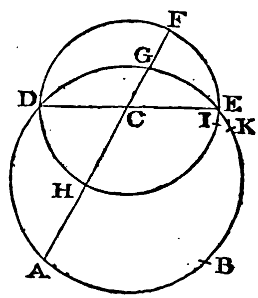
## PROP. XI. PROB.

Through two given points, to describe a circle bisecting the circumference of a given circle.

Let A and B be two points, through which it is required to describe a circle ADGEB, that shall bisect the circumference of the circle HDFE.

### ANALYSIS.

Join D, E, the points of intersection. Because DFE is, by hypothesis, a semicircumference, DE is a diameter, and must, therefore, pass through the centre C. Join AC, and produce it to F. Since DC=CE, it is evident that $AC.CG = DC^2 = HC.CF$; but the rectangle HC, CF is given, and consequently the rectangle AC, CG is also given; and AC being given, CG is hence given, and the point G. Wherefore the three points A, G, and B, being given, the circle AGB is given.




### COMPOSITION.

Through C, the centre of the given circle, draw ACF, make AC : HC : : CF or HC : CG, and through the three points A, G, and B, describe the circle AGB: This will bisect the circumference HDFE.

For, through one of the points of intersection, draw the diameter DCI, and produce it to meet the circumference of the circle AGB in K. Because AC : HC : : HC : CG, the square of HC is equal to the rectangle AC and CG; but $HC^2 = DC.CI$, and $AC.CG = DC.CK$: wherefore $DC.CI = DC.CK$, and $CI = CK$, or the points I and K are one, and the circle AGB passes through both extremities of the diameter of HDFE.

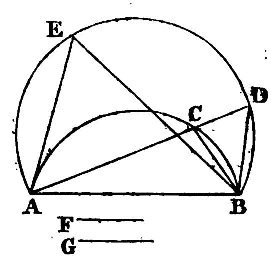
## PROP. XII. PROB.

Two arcs of different circles being constituted on the same chord, to draw from its extremity a straight line, of which the portion intercepted by those arcs shall be equal to a given line.

Let the arcs ACB and AEB stand on the same base AB; it is required to draw ACD, such that CD shall be equal to F.

### ANALYSIS.

Join BC and BD. The angle CDB contained in the arc AEB is given; and so is ACB, included in the smaller arc, and consequently its supplement BCD; wherefore all the angles of the triangle CBD are given, and the triangle itself is given in species. But it is likewise given in magnitude, for the side CD is given. Hence BD is given, and therefore the point D and the line ACD are also given.




### COMPOSITION.

Apply the tangent AE to the inner arc, and join BE; make AE to BE as F to G, and in the exterior segment inflect G from B to D; ACD being drawn in the line required.

For join BC and BD, the angle AEB is equal to ADB or CDB, and the angle EAB is equal to the angle in the segment alternate to ACB, or to the exterior angle BCD; consequently the triangle AEB is similar to CDB, and

AE : BE : : CD : BD; but, by construction, AE : BE : : F : G, and hence CD : BD : : F : G; wherefore since BD was made equal to G, CD must be equal to F.

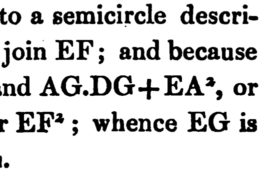
## PROP. XIII. PROB.

To cut a given straight line, such that the square of one part shall be equivalent to the rectangle under the remainder and another given straight line.

Let AB be a straight line, from which it is required to cut off a segment AG, whose square shall be equivalent to the rectangle under the remainder and the straight line C.

### ANALYSIS.

Produce BA till AD be equal to C, on DB describe a semicircle and erect the perpendicular AF. Because $AG^2 = C \times GB$, it follows that DA : AG : : AG : GB; wherefore DA : AG : : DG : AB, and consequently DA.AB $=$ AG.DG; but DA.AB $= AF^2$; and therefore AG.DG $= AF^2$; whence AF is equal to a tangent drawn from G to a semicircle described on DA. Bisect DA in E, and join EF; and because AG.DG $= AF^2$, add $EA^2$ to each, and AG.DG $+ EA^2$, or $EG^2$, is equivalent to $AF^2 + EA^2$ or $EF^2$; whence EG is equal to EF, and is therefore given.




### COMPOSITION.

Having produced AD equal to C, and described on BD a semicircle, erect the perpendicular AF, bisect AD in E, join EF and make EG equal to it; the square of the seg-

ment AG thus formed in AB is equivalent to the rectangle under the remaining part GB and the given line C.

For EFA being a right angled triangle $EF^2 = EA^2 + AF^2$, and consequently $AF^2 = EF^2 - EA^2$, or $EG^2 - EA^2$; and since $EG^2 - EA^2 = (EG + EA)(EG - EA)$, or $DG.AG$, therefore $AF^2 = DG.AG$. But $AF^2 = DA.AB$; whence $DG.AG = DA.AB$, and $AG : AB :: DA : DG$; wherefore $AB - AG$, or $GB : AG :: DG - DA$ or $AG : DA$, whence $AG^2 = GB.DA$.

If DA, or C, be equal to AB, then $AG^2 = AB.BG$, and hence $AB : AG :: AG : BG$, so that the line AB is now divided in extreme and mean ratio at the point G. The construction becomes evidently the same with that given for the medial section of a line, (II. 15.) which was indeed only a simple case of this problem.

## PROP. XIV. PROB.

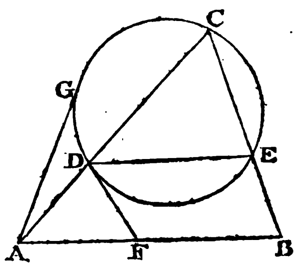
Through two given points, to draw straight lines to a point in the circumference of a given circle, so that the chord of the intercepted segment shall be parallel to the straight line which joins the given points.

Let it be required, from the points A and B, to inflect AC and BC cutting the given circumference in D and E, such that the chord DE shall be parallel to AB.

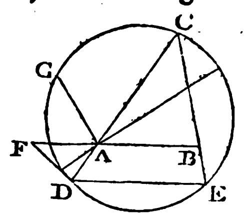
### ANALYSIS.

From D draw the tangent DF, meeting AB in F. The angle FDE is equal to the angle ECD, or its supplement, in the alternate segment; but DE being parallel to AB,

the angle FDE or its supplement is equal to the alternate angle AFD, which is consequently equal to the angle ECD or ACB; wherefore the triangles ADF and ABC, having likewise a common angle CAB, are similar, and AD : AF : : AB : AC, and hence




AD.AC=AF.AB. But since the point A and the circle DCE are given, the rectangle AD, AC is also given; for it is equivalent to the square of the tangent AG, when A lies without the circumference,—and equivalent to the square of AG, a perpendicular to the diameter, in the case where




that point lies within the circle. Hence the rectangle AF, AB is given; and AB being given, AF is likewise given, and consequently the point F. Wherefore the tangent FD is given in position; and since the point A is given, the straight line AC is given, and thence BC and the intersection E.

## COMPOSITION.

If the point A be without the circle, draw the tangent AG; or if it lie within the circle, erect AG perpendicular to the diameter which passes through it. Make AB : AG : : AG : AF, from F draw the tangent FD, join AD, and produce it to meet the opposite circumference in C, join CB, cutting the circle in E; the straight line DE is parallel to AB.

For, since AB : AG : : AG : AF, $AG^2$=AB.AF; but $AG^2$=CA.AD, whence AB.AF=CA.AD, and conse-

quently AB : AC : : AD : AF. Wherefore the triangles BAC and DAF, having the sides about their common angle proportional, are similar, and hence the angle ACB is equal to AFD; but ACB or DCE is equal to EDF or its supplement, and consequently the angle AFD is equal to EDF or its supplement, and the chord DE is parallel to AB.

## PROP. XV. PROB.

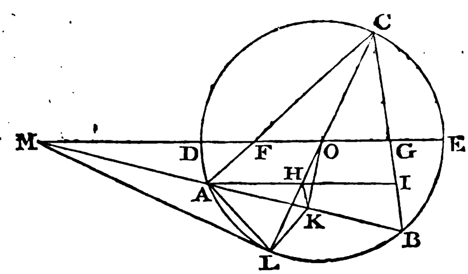
From two given points in the circumference of a given circle, to inflect straight lines to another point in the opposite circumference, such as to intercept, on either side of the centre, equal segments of a given diameter.

Let it be required from the points A and B, to inflect AC and BC, so as to intercept, on the diameter DE, equal portions from the centre.

### ANALYSIS.

Join BA, and produce it and the diameter ED to meet in M, draw COL, from O let fall the perpendicular OK upon AB, join LK, through A draw AHI parallel to DE, and join HK.

The parallels FG and AI are cut proportionally by the diverging lines CA, CH, and CI; but FO is equal to OG, and consequently AH is equal to HI. Wherefore HK is parallel to IB, and the angle AKH is equal to ABI; and



since the angle ABI or ABC is equal to ALC, the angle AKH is equal to ALC or ALH, and hence the quadrilateral figure AHKL is contained in a circle. Consequently the angle HAK is equal to HLK; but HAK is equal to OMK, which is therefore equal to HLK or OLK, and thence the quadrilateral figure MOKL is also contained in a circle. Wherefore the angle MLO is equal to MKO; but MKO is a right angle, and consequently MLO is likewise a right angle, and thence ML is a tangent. But the point M, being the concourse of ED and BA, is given, and therefore the tangent ML to the given circle is given; whence the diameter LC, and the point C, are given.

## COMPOSITION.

Produce ED and BA to meet in M, draw the tangent ML and the diameter LC; the straight lines AC and BC will cut off from the centre equal portions, OF and OG, of the given diameter ED.

For draw AI parallel to DE, and OK perpendicular to AB, and join LK and KH.

Because ML is a tangent, MLO is a right angle, and, therefore, equal to MKO; consequently MKL is equal to MOL, that is, to AHL. Wherefore the quadrilateral figure AHKL is contained in a circle, and hence the angle ALH is equal to AKH; but, for the same reason, ALH or ALC is equal to ABC or ABI, and consequently AKH is equal to ABI, and KH parallel to BI. Now since AK is equal to KB, it follows that AH is equal to HI, and hence that FO is equal to OG.

## PROP. XVI. PROB.

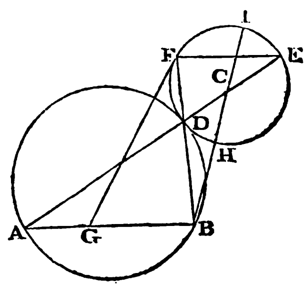
Through two given points, to describe a circle touching a given circle.

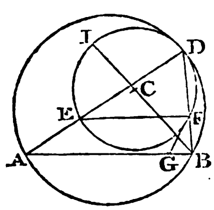
Let it be required, through the points A and B, to describe a circle, touching another circle whose centre is C.

### ANALYSIS.

Through D, the point of contact, draw ADE and BDF, join EF, at F apply the tangent FG, and draw BHCI.

Because FG touches the given circle, the angle BFG is equal to FED, and therefore equal to BAD, since FE and AB are parallel; but the triangles BGF and BDA have likewise a common angle at B, and are hence similar; therefore BF : BG : : BA : BD, and BA.BG = BF.BD = BI.BH. But BI and BH are given, and thence the rectangle BA, BG is given, and consequently the point G is given. Hence the tangent GF, and D, the intersection of BF, are given; wherefore the circle which passes through the three points A, D, and B, is given.




### COMPOSITION.

Make BA : BI : : BH : BG, draw the tangent GF, join BF cutting the given circumference in D, and through

the points A, D, and B, describe a circle; this will touch the circle FDE.

For draw ADE, join FE, and draw BHCI. Since BA : BI : : BH : BG, therefore BA.BG=BI.BH= BF.BD; whence BF : BG : : BA : BD, and consequently the triangles BGF and BDA, having the same vertical angle, are similar, and hence the angle BFG is equal to BAD. But BFG is equal to FED, and thus the alternate angles BAE and FEA are equal, and FE is parallel to AB; whence the two circles touch at D.

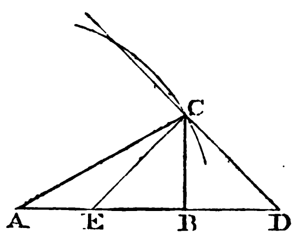
## PROP. XVII. PROB.

Given the hypotenuse of a right-angled triangle, and the sum or difference of the base and perpendicular, to construct the triangle.

### ANALYSIS.

In the base AB, or its production, make BD or BE equal to the perpendicular BC, and join CD and CE.

The triangles CBD and CBE are right-angled and isosceles, and therefore the angles at D and E are each of them half a right angle. If AD, the sum of AB and BC, be given, the point D is given, and consequently the straight line DC, making a given angle with DA, is given in position; or if AE, the difference between the base and perpendicular, be given, the point E is given, and the straight line EC is given in position. But the hypotenuse AC being given, the



point C must, therefore, occur in the contact or intersection of a circle described from A with that radius and the straight line CD or CE. Consequently C is given, the perpendicular CB, and thence the right-angled triangle ABC.

## COMPOSITION.

Make AD or AE equal to the sum or difference of AB and BC; draw DC or EC at an angle CDE or CED equal to half a right angle; from A with the radius AC describe a circle meeting DC or EC in the point C, and from C let fall the perpendicular CB: ACB is the triangle required.

For the right-angled triangles CBD and CBE are evidently isosceles, and therefore AD is equal to the sum, and AE to the difference, of AB and BC.

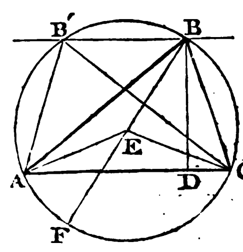
## PROP. XVIII. PROB.

Given the base of a triangle, its altitude, and the rectangle under its two sides,—to determine the triangle.

## ANALYSIS.

About the triangle ABC describe a circle, and draw the diameter BF and the radii AE and CE.

Because the given rectangle AB.BC is equivalent to BD.BF, this rectangle is likewise given; and since the perpendicular BD is given, the diameter BF, and therefore the radii AE, CE, are given. But the base AC being given, the triangle AEC is hence given, and consequently the centre E and the cir-



cle ABCF are given. Again, because BD, the distance of the vertex of the triangle from its base, is given, that point must occur in the parallel BB′, and, being thus placed in the contact or intersection of a given straight line with a given circle, is itself given.

### COMPOSITION.

On BD construct a rectangle equivalent to the given space; also form on AC the triangle AEC, having AE and CE each equal to half the greater side of that rectangle; from E with the radius EA describe a circle; on AC erect a perpendicular DB equal to the altitude of the triangle, and through B draw a parallel meeting the circumference in B or B′: ABC is the triangle required.

For ABC has evidently the given altitude BD, and the rectangle AB.BC, being equivalent to BF.BD, is therefore equivalent to the given space.

## PROP. XIX. PROB.

To inscribe in a circle a quadrilateral figure, of which the sides are given.

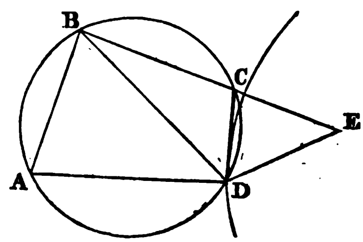
Let it be required to find a circle that will circumscribe the quadrilateral figure ABCD, contained by given sides.

### ANALYSIS.

Join the diagonal BD, from D draw DE making with the side CD an angle CDE equal to ABD, and meeting BC produced in E.

The exterior angle DCE of the inscribed quadrilateral is equal to BAD, and the angle DCE is by construction

equal to ABD; wherefore the triangles ADB and DEC are similar, and AB : AD : : CD : CE; but the three terms of this analogy being given, the fourth is also given and thence the point E. Again, from the similitude of the same triangles, AB : BD : : CD : DE, and alternately AB : CD : : BD : DE; but AB and CD being both given,




their ratio is given, and consequently that of BD to ED. The points B and E being thus given, and likewise the ratio of the inflected lines BD and ED, the *locus* of their concourse D is, by the tenth Proposition of this book, a given circle. But CD is given, and therefore the point D, and the three points B, C, and D, being all given, the circle which passes through them is hence given.

## COMPOSITION.

Let L, M, N, and O express in succession the several sides of the quadrilateral figure; make L : M : : N : CE, and annex this in the same straight line with the fourth side O or BC; describe, by Prop. 10. a circle, which is the *locus* of lines inflected from B and E in the ratio of L to N; from C inflect to that circle CD equal to N; about the three points B, C, and D, describe another circle, in which, from B, insert AB equal to L; and AD, being joined, will be equal to M. For join BD and ED. Since, from the property of *loci*, AB : CD : : BD : DE, alternately AB : BD : : CD : DE, and the angle ECD being equal to BAD, the triangles ABD and CDE are similar, and AB : AD : : CD : CE; but by construction

AB : M : : CD : CE, and consequently AD : M : : CD : CD, and AD is equal to M. The circle described admits, therefore, all the four chords which form the sides of the quadrilateral figure.

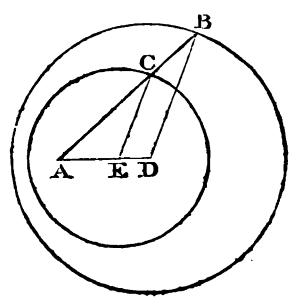
## PROP. XX. THEOR.

If a straight line, drawn through a given point to the circumference of a given circle, be divided in a given ratio, the *locus* of the point of section will also be the circumference of a given circle.

Let AB, terminating in a given circumference, be cut in a given ratio; the segment AC will likewise terminate in a given circumference.

### ANALYSIS.

Join A with D the centre of the given circle, and draw CE parallel to BD. It is obvious that AC : AB : : AE : AD; whence the ratio of AE to AD being given, AE and the point E are given. Again, since AC : AB : : CE : BD, the ratio of CE to BD is given, and consequently CE is given in magnitude. Wherefore the one extremity E being given, the other extremity of CE must trace the circumference of a given circle.




### COMPOSITION.

Join AD, and divide it at E in the given ratio, and in the same ratio make DB to the radius EC, with which, and from the centre E, describe a circle.

For draw AB cutting both circumferences, and join CE and BD. Because CE : BD : : AE : AD, alternately CE : AE : : BD : AD; wherefore the triangles CAE and BAD, having likewise a common angle, are similar, and consequently AC : CB : : AE : AD, that is, in the given ratio.

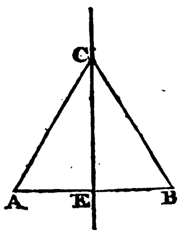
## PROP. XXI. THEOR.

**If, from two given points, there be inflected two straight lines in a given ratio, the *locus* of their point of concourse is a straight line, or a circle given in position.**

Let AC and BC, drawn from the points A and B, have a given ratio; then will C, the point of concourse, lie in a straight line given in position, or in the circumference of a given circle.

1. *When the inflected lines are equal, they terminate in a straight line given in position.*

### ANALYSIS.




Bisect AB in E, and join EC. The triangles ACE and BCE, having the sides AE and AC equal to BE and BC, and EC common, are equal; wherefore the angle AEC is equal to BEC, and EC is perpendicular to AB, and consequently given in position.

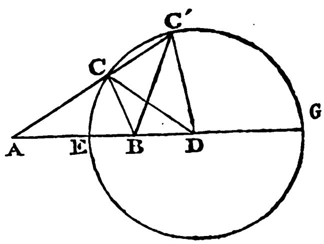
### COMPOSITION.

Bisect AB by the perpendicular EC, which is the *locus* required. For draw AC and BC to any point in it, and the triangles AEC and BEC are evidently equal, and hence AC is equal to BC.

2. *When the inflected lines AC and BC have a ratio of inequality, their point of concourse lies in the circumference of a given circle.*

## ANALYSIS.

Draw CD, making the angle BCD equal to BAC, and meeting AB produced in D. The triangles DAC and DCB, having the angle at D common, and the angles at A and C equal, are evidently similar; and hence AD : AC : : DC : BC, and alternately AD : DC : : AC : BC, that is, in the given ratio. But AD : DC : : DC : BD, and consequently AD is to BD in the duplicate of the given ratio of AD to DC, and which is therefore likewise given.




Consequently BD, and the point D, are given; and DC being thence given, its extremity C must lie in the circumference of a circle described with that radius.

## COMPOSITION.

Divide AB in the given ratio at E, and in the same ratio make ED to BD; the circle described from the centre D, and with the radius DE, is the *locus* required.

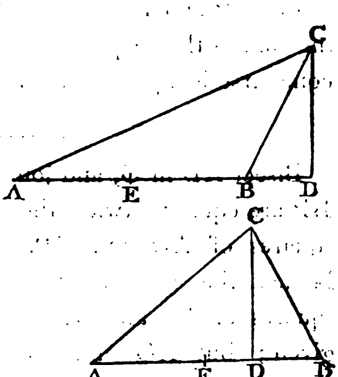
For, since AE : BE : : ED : BD, it follows that AD : ED or DC : : ED or DC : BD; hence the triangles DAC and DCB, having the sides which contain their common angle at D proportional, are similar, and therefore AC : AD : : BC : DC, or alternately AC : BC : : AD : DC or ED, that is, in the given ratio.

## PROP. XXII. THEOR.

**If from two given points there be inflected two straight lines, of whose squares the difference is given, the *locus* of their point of concourse will be a straight line given in position.**

Let AC and BC, drawn from the points A and B, have the difference of their squares given; the *locus* of C, the point of concourse, is a straight line given in position.

### ANALYSIS.




Draw CD perpendicular to AB, which bisect in E. The difference between the squares of AC and BC is equal to twice the rectangle under AB and ED; consequently that rectangle, and its containing side ED, are given; whence the point of bisection E being given, the point D is given, and the perpendicular CD is therefore given in position.

### COMPOSITION.

Bisect AB in E, and make the rectangle under twice AB and ED equal to the given space; the perpendicular DC is the *locus* required.

For $AC^2 - BC^2 = AB.2ED = 2AB.ED$, and conse-

quently the difference of the squares of AC and BC is equal to the given space.

## PROP. XXIII. THEOR.

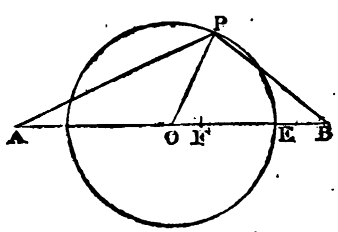
**If, from given points, there be inflected straight lines, whose squares are together equal to a given space,—their point of concourse will terminate in the circumference of a given circle.**

1. *When there are only two given points.*

Let AP and BP, drawn from the points A and B, have the sum of their squares given; the *locus* of their point of concourse is a given circle.

### ANALYSIS.

Bisect AB in O, and join OP. The squares of AP and BP are equal to twice the squares of AO and OP. Hence the sum of the squares of AO and OP is given; but AO and its square being given, the square of OP and OP itself must be given; wherefore the *locus* of the extremity P is a circle, of which the point of bisection is the centre.




### COMPOSITION.

Bisect AB in O; find AF the side of a square equal to half the given space, and make $OE^2 = AF^2 - AO^2$; the point O is the centre, and OE the radius, of the required circle.

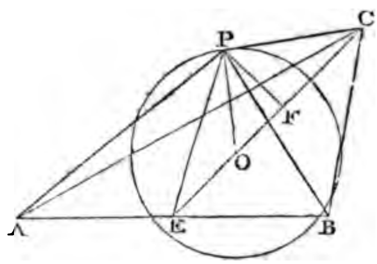
For $AP^2 + BP^2 = 2AO^2 + 2OP^2 = 2AO^2 + 2OE^2 = 2AF^2$, or the given space.

2. *When three points are given.*

Let the straight lines AP, BP and CP, inflected from the points A, B, and C, have the sum of their squares given; the *locus* of their point of concourse is a given circle.

## ANALYSIS.

Bisect AB in E, and $AP^2+BP^2=2AE^2+2EP^2$; consequently $AP^2 + BP^2 + CP^2 = 2AE^2 + 2EP^2 + CP^2$. Now $2AE^2 = AB.BE$, and, letting fall the perpendicular PF, $2EP^2 = 2EF^2 + 2PF^2$, and $CP^2 = PF^2 + CF^2$.




Wherefore $AP^2+BP^2+CP^2=AB.BE+3PF^2+2EF^2+CF^2$. Trisect EC in the point O, and join PO; and, $2EF^2+CF^2=EC.CO+3OF^2$ *. Whence $AP^2+BP^2+CP^2 = AB.BE + EC.CO + 3PF^2 + 3OF^2 = AB.BE + EC.CO + 3PO^2$. But the intermediate points of division E and O are evidently given, and thence the rectangles AB, BE and EC, CO, are given; wherefore $3PO^2$ is given, and consequently PO itself. Since one extremity of that line then is given, the other extremity P must lie in the circumference of a given circle, having PO for its radius.

* For $EF^2=EO^2+OF^2+2EO.OF$, (or CO.OF), and $2EF^2=2EO^2$, (or CO.EO), $+ 2OF^2+2CO.OF$; but $CF^2 = CO^2+OF^2-2CO.OF$, and consequently $2EF^2+CF^2=CO\ (EO+CO)$, or $CO.CE+3OF^2$.

## COMPOSITION.

Bisect AB in E, trisect EC in O, and find OP such that its square shall be triple the excess of the given space above the rectangles AB, BE and EC, CO; the *locus* required is a circle, of which O is the centre, and PO the radius. For $3PO^2=3PF^2+3OF^2$, $3PO^2+EC.CO=3PF^2+EC.CO+3OF^2=3PF^2+2EF^2+CF^2=2PE^2+PF^2+CF^2=2PE^2+CP^2$; consequently the given space, or $3PO^2+AB.BE+EC.CO=2AE^2+2PE^2+CP^2=AP^2+BP^2+CP^2$.

By pursuing this mode of investigation, the proposition may be successively extended to any number of points, by the quadrisection and quinquisection, &c. of the lines drawn to the fourth and fifth, &c.

## PROP. XXIV. PROB.

In a straight line given in position, to find a point, whose distances from two given points on the same side shall together be the least possible.

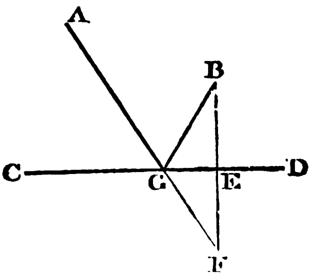
Let it be required, from the points A and B to some point in CD, to draw the lines AG and BG, forming together a *minimum*.

## ANALYSIS.

From B, either of the given points, let fall the perpendicular BE upon CD, and, having produced it equally on the opposite side, join GF. It is obvious that the triangles BEG, FEG are equal, and consequently that BG=GF; whence AG + GF is a *minimum*. But the points A and F are evidently both given, and since the shortest communication between them is a straight line, its intersection G with CD is given, and therefore the inflected lines AG

and BG are given in position.

*Cor.* It hence appears that, when the combined distance of the points A and B from the straight line CD is the least possible, the incident angles AGC and BGD are equal.

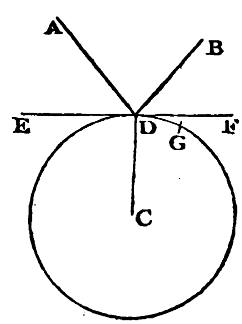



## PROP. XXV. THEOR.

Straight lines drawn from two given points to the circumference of a given circle are the least possible, when they make equal angles with a tangent applied at the point of inflection.

Of all the straight lines inflected from the points A and B to the circumference of the circle GDH, AD and BD, which meet the tangent EF at equal angles, form together a *minimum*.

For, by the last proposition, AD and BD, falling at an equal incidence, are jointly shorter than any other lines inflected from the points A and B to the straight line EF; but such lines drawn to that tangent are less than the exterior lines which terminate in the circumference; whence, for both these reasons combined, AD and BD must form the *minimum*



of all the straight lines inflected to the circumference GDH.

*Cor.* It may be shewn nearly in the same way, that the lines from A and B attain their *maximum*, when they are inflected to a point where DC meets the opposite concave circumference.

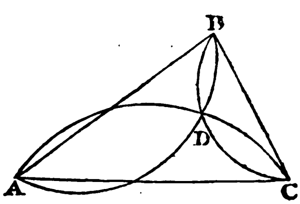
## PROP. XXVI. PROB.

To find a point, whose distances from three given points are the least possible.

Let it be required, from the points A, B, and E, to draw AD, BD, and CD, such that their sum shall be a *minimum.*

### ANALYSIS.

If the distance BD were supposed to remain constant, the position of D, in the circumference of a circle described from B with the radius BD, must, by the last proposition, be such, when AD and CD together compose a *minimum*, that the angle ADB shall be equal to CDB. For the same reason, if AD continued invariable, BD and CD, completing the *minimum*, must form with it equal angles ADB and ADC. Whence, uniting these conditions, the straight lines AD, BD, and CD all make equal angles about their point of concourse.




It is hence obvious, that the point D must occur in the intersection of the arcs of circles described about equilateral triangles, constituted externally on the sides AB, BC, or AC.

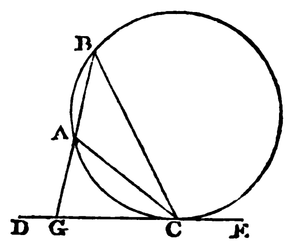
## PROP. XXVII. PROB.

**In a straight line given in position, to find a point, at which the straight lines, drawn to two given points on the same side, shall contain the greatest angle.**

Let it be required to draw AC and BC to a point C in DE, so that the angle ACB shall be a *maximum.*

### ANALYSIS.

Describe a circle about the points C, A, and B. Because the angle ACB is greater than any other which has its vertex in DE, the circumference must lie within that straight line, and therefore DE touches the circle.




It hence follows, that $GA.GB = GC^2$; and, therefore, the point C is assigned, by finding GC a mean proportional to GA and GB.

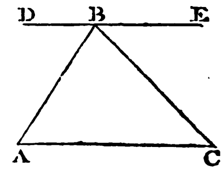
## PROP. XXVIII. PROB.

**To find a triangle with a given perimeter, and standing on a given base, that shall contain the greatest area.**

Let it be required to find a triangle ABC, constituted on the base AC, and containing, within a given perimeter, the greatest possible surface.

## ANALYSIS.

Since the base of the triangle ABC is constant while its area forms a *maximum*, the corresponding altitude must evidently be the greatest possible, and consequently the vertex B must lie in a parallel the remotest from AC. Supposing, therefore, the parallel DE to retain its place, the sum of the sides AB and CB, and consequently the whole perimeter of the triangle, will, by the twenty-fourth Proposition of this Book, be the least possible, when the angle ABD is equal to CBE. Whence, preserving the same perimeter, the parallel will be enabled to recede to the greatest distance from AC, if these incident angles still maintain their equality; but DE being parallel to AC, the alternate angles BAC and BCA are likewise equal, and consequently their opposite sides CB and AB. The triangle ABC is thus isosceles; and it is also given, for its sides are all given.



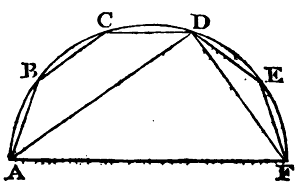

*Cor.* Hence an equilateral polygon, under a given number of sides, contains, within the same perimeter, the greatest possible surface: For, the rest of the figure remaining constant, suppose any two adjacent sides to vary, and the accrescent triangle so formed will, by this proposition, be a *maximum*, when those sides are equal. The polygon, deriving its expansion from the aggregate of the exterior triangles, must therefore be the greatest possible, when such triangles are in every combination isosceles, and consequently when all the sides of the figure equal.

## PROP. XXIX. THEOR.

If a polygon have all its sides given, except one, it will contain the greatest area, when it can be inscribed in a semicircle, of which this indeterminate side is the diameter.

Let the polygon ABCDEF, having given sides AB, BC, CD, DE and EF, stand upon a base AF, which is variable; the area will attain its *maximum*, when AF becomes the diameter of a circumscribing semicircle.

For AD and FD being inflected to any point D, the spaces ABCD and DEF will evidently remain the same, while the angle ADF is enlarged, or the points A and F are distended. Whence the polygon must contain the greatest area, when the included triangle ADF, contained by given sides AD and DF, is a *maximum*. Now, this will take place when the altitude of the triangle, or the perpendicular let fall from the vertex F upon AD, is the greatest possible. Wherefore ADF is a right angle, and consequently the point D lies in a semicircumference. But the same reason applies to every other intermediate point B, C, or E, of the polygon, which consequently, in its state of *maximum*, is disposed within a semicircle described on the variable side AF.




*Cor.* 1. Hence a polygon, whose sides are all given, contains the greatest area, when it can be inscribed in a circle.

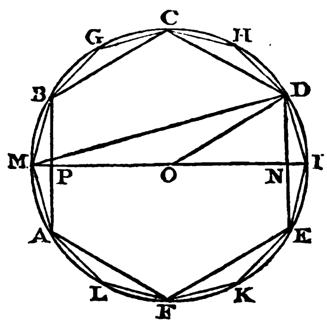
For let ABCD be a polygon, which has each of its sides AB, BC, CD, and AD given. Draw the diameter AF, and join DF. The polygon ABCDF is thus a *maximum*; but the triangle ADF being evidently determinate, the remaining polygon ABCD is likewise a *maximum*.

*Cor.* 2. Hence a regular polygon, with a given perimeter, and formed by a given number of sides, contains the greatest area. For, by the corollary to the last Proposition, the sides are all equal; but its angles are also equal, since it occupies the circumference of a circle.

## PROP. XXX. THEOR.

A circle contains, within a given perimeter, the greatest possible area.

From the preceding investigations, it appears, that the perimeter and number of sides being given, the figure of greatest capacity is a regular polygon. Let ABCDEF be such a polygon, bounded by the given perimeter: Bisect the corresponding arcs of the circumscribing circle, and another regular polygon




MBGCHDIEKFLA will arise, having twice the number of sides. Draw the diameter MI, and join MD and OD. Both polygons are alike composed of triangles equal to ODN and ODI, and consequently the area of the polygon ABCDEF is to that of MBGCHDIEKFLA as ON

to OI, or as 2ON or PN to 2OI or MI. But if this exterior polygon MBGCHDIEKFLA were contracted to the same perimeter with ABCDEF, its area would be diminished in the ratio of $DI^2$ to $DN^2$, that is, in the ratio of the rectangle MI, NI to MN, NI, or that of MI to MN. Whence the original polygon is to another of equal perimeter and with double the number of sides, as PN to MN. An isoperimetrical figure thus has it area always increased, by doubling the number of its sides. Continuing this duplication, therefore, the regular polygons which arise in succession will have their capacity perpetually enlarged. Whence the circle, as it forms the limit or extreme boundary of all those polygons, must, with a given circumference, contain the greatest possible space.

# PRINCIPLES

OF

# PLANE TRIGONOMETRY.

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TRIGONOMETRY is the science of calculating the sides or angles of a triangle. It combines the principles of Geometry and Arithmetic.

The sides of a triangle are measured, by referring them to some definite portion of linear extent, which is fixed by convention. The mensuration of angles is effected, by means of that universal standard derived from the partition of a circuit. Since angles are proportional to the intercepted arcs of a circle described from their vertex, the subdivision of the circumference must determine their magnitude. A quadrant, or the fourth part of the circumference, corresponding to a right angle, hence forms the basis of angular measures. But these measures depend on the relation of certain classes of lines connected with the circle, which it is necessary previously to investigate.

## DEFINITIONS.

1. The *complement* of an arc is its defect from a quadrant; its *supplement* is its defect from a semicircumference; and its *explement* is its defect from the whole circumference.

2. The *sine* of an arc is a perpendicular let fall from one of its extremities upon a diameter passing through the other.

3. The *versed sine* of an arc is the portion of a diameter intercepted between its sine and the circumference.

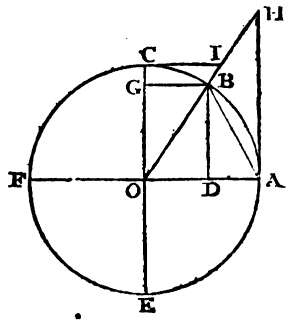
4. The *tangent* of an arc is a perpendicular drawn at one extremity to a diameter, and limited by a diameter extending through the other.

5. The *secant* of an arc is the straight line which joins the centre with the termination of the tangent.

In naming the *sine*, or *tangent*, of the *complement* of an arc, it is usual to employ the abbreviated terms of *cosine*, *cotangent* and *cosecant*. A farther contraction is frequently made in noting the radius and other lines connected with the circle, by retaining only the first syllable of the word, or even the mere initial letter.

Let ACFE be a circle, of which the diameters AF and CE are at right angles; having taken any arc AB, produce the radius OB, and draw BD, AH perpendicular to AF, and BG, CI per-

pendicular to CE. Of this assumed arc AB, the *complement* is BC, the *supplement* is BCF, and the *explement* is BCFEA; the *sine* is BD, the *cosine* BG or OD, the *versed sine* AD, the *coversed sine* CG, and the *supplemental versed sine* FD; the *tangent* of AB is AH, and its *cotangent* CI; and the *secant* of the same arc is OH, and its *cosecant* OI.




Several obvious consequences flow from these definitions:

1. Since the diameter which bisects an arc bisects also the chord at right angles, it follows that half the chord of any arc is equal to the sine of half that arc.

2. In the right-angled-triangle ODB, $BD^2+OD^2=OB^2$; and hence the squares of the sine and cosine of an arc are together equal to the square of the radius.

3. The triangle ODB being evidently similar to OAH, OD : DB : : OA : AH; that is, the cosine of an arc is to the sine, as the radius to the tangent.

4. From the similar triangles ODB and OAH, OD : OB : : OA : OH; wherefore the radius is a mean proportional between the cosine and the secant of an arc.

5. Since $BD^2=AD.FD$, it is evident that the sine of an arc is a mean proportional between the versed sine and the

supplemental versed sine, or between the sum and difference of the radius and the cosine.

6. Hence also the chord of an arc is a mean proportional between the versed sine and the diameter; for $AB^2 = AD.AF$.

7. The triangles OAH and ICO being similar, AH : OA : : OC : CI; and hence the radius is a mean proportional between the tangent of an arc and its cotangent.

8. Since $OD^2 = BG^2 = CG.CE$, it follows that the cosine of an arc is a mean proportional between the sum and difference of the radius and the sine.

In the first semicircle, the sines may be considered as positive or additive; but in the second semicircle as negative or subtractive.

Again, the cosine BG or DO in the first and third quadrants is subtractive, and in the second and fourth, additive; but the versed sine AD is subtractive through the whole circuit, increasing till it equals the diameter AF in the upper semicircle, and again decreasing in the under semicircle till it vanishes.

To perceive more clearly the connexion of those lines derived from the circle, we may trace their successive values while the corresponding arc is continually increased. In the first quadrant AC, the sine BD augments till it constantly equals the radius CO; but in the second quadrant CF, it again diminishes and becomes extinct at F.

In the third and fourth quadrants FE and EA, the sine reappears on the opposite side, increasing, and afterwards decreasing till it vanishes at A.

The tangent AH continually augments in the first quadrant, but in the second quadrant it reappears on the opposite side, diminishing till it becomes extinct. In the third and fourth quadrants the same order of changes is repeated. In the first and third quadrants, therefore, the tangent is additive, but in the second and fourth quadrants subtractive. The cotangent has the same character.

The circumference of the circle is commonly divided into 360 equal parts, called degrees, each of them being subdivided into 60 minutes, and these again being each distinguished into 60 seconds. It very seldom is required to carry this subdivision any further. Degrees, minutes, seconds, or thirds, are conveniently noted by these marks, $^{\circ}\ '\ ''\ '''$. Thus, $23^{\circ}\ 27'\ 33''\ 42'''$, signifies 23 degrees, 27 minutes, 33 seconds, and 42 thirds.

As grounds of trigonometrical calculation, it is requisite to form tables of the length of the several lines connected with the circle. The radius being reckoned unit, those measures are referred to that standard, and commonly carried to seven decimal places.

The SINES corresponding to every arc of the Quadrant may be computed from this

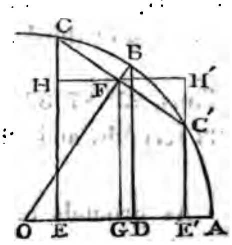
## PROPOSITION.

Of three equidifferent arcs, the rectangle under the radius and the sum of the sines of the extremes is equal to twice the rectangle under the cosine of the common difference and the sine of the mean arc.

Let A—B, A, and A+B represent three arcs, increasing by the common difference B; then

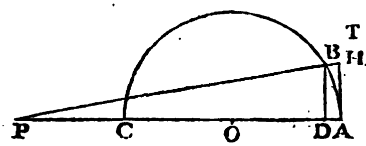
$$R\big(sin(A+B)+sin(A-B)\big)=2cosB\ sinA.$$

If AB represent the mean arc, and BC or BC′ the common difference, the three arcs will be AC′, AB and AC. Join OB and CC′, intersecting in F, through which draw FC and HFH′ parallel to BD and OA. It is evident, that the radius OB bisects CFC′ at right angles, and that the opposite sides CH and C′H′ of the right-angled triangles CHF and C′H′F are equal. But, from the property of parallel and diverging lines, OB : OF : : BD : FG, and consequently OB : 2OF : : BD : 2FG, or C′E′+CE; wherefore OB (C′D′+CD) =2OF.BD.




If the point C′ coincide with A, then OB.CD = 2OF.BD; or $R.sin2A=2cosA\,sinA$.

Since a small arc AB is evidently less than its tangent AT, and greater than its sine BD, it will be nearly equal to a portion AH of the tangent, as limited by a straight line PBH, drawn from some point P in the diameter, or its extension beyond the centre O. To find this point, assume an arc AB of 18°, or the twentieth part of the circumference, and therefore denoted by .31416; the corresponding sine BD, being half the side of an inscribed decagon, is therefore approximately .30902, whence the cosine OD is computed to be .95106. But, from similar triangles, AH : BD : : AP : DP, and consequently AH—BD : AH : : AP—DP : AP; that is, .00514 : .31416 : : .04894 : 2.9903, and hence AP may be always taken as equal to three times the radius OA, the point P thus lying without the circle, and at a distance equal to the radius.




Now, the ratio of PD to PA, as it approaches to that of equality, will not be sensibly affected, by taking from each of the terms small equal differences. Whence BD is to AH, or the arc AB, very nearly as PD—2DA to PA—DA, or 3OD to 2OD+OA; but OD : OA : : BD : AT, and therefore BD : AH : : 3BD : 2BD+AT; and the third term of this analogy being triple of the first, the fourth term must likewise be triple of the second, that is, twice the sine of the arc AB, together with its tangent, approximates to three times the length of the arc.

Again, the small arc, its sine and tangent, being denoted respectively by $a$, $s$, and $t$, since $BD^2 = CD.DA$, it follows that the versed sine DA must be very nearly equal

to $\frac{s^2}{2}$. Wherefore $s : t :: 1 : 1 + \frac{s^2}{2}$, and consequently $t = s + \frac{s^2}{2}$ and $3a = 3s + \frac{s^2}{2}$, or $a = s + \frac{s^2}{6}$.

Since the circumference of a circle whose radius is unit was found (Prop. 20, Book V.) to be 3.14159265536, the 360th part of this, or the arc of one degree, is equal to .0174533, it follows from what has been just shown, that the sine of $1° = .0174533 - \frac{1}{6}(.0174533)^3 = .0174524$, and hence the versed sine of $1° = \frac{1}{2}(.0174524)^2 = .0001523$. Wherefore $sin(A + 1°) = 2sin A - 2sin A \times .0001523 - sin(A - 1°)$; or, *if from twice the sine of any arc, diminished by its product, into .0001523, the sine of an arc one degree lower be subtracted, the remainder will exhibit the sine of an arc which is one degree higher.* Thus,

$Sin\,2° = 2sin\,1° - 2sin\,1° \times .0001523 = .0349048 - .0000053$
$= .0348995.$

$Sin\,3° = 2sin\,2° - 2sin\,2° \times .0001523 - sin\,1° = .0697990 -$
$.0000106 - .0174524 = .0523360.$

$Sin\,4° = 2sin\,3° - 2sin\,3° \times .0001523 - sin\,2° = .1046720 -$
$.0000160 - .0348995 = .0697565.$

To fill up the sines of the minutes, it must be observed, that the arc or sine of one minute being .0002909, its versed sine is $\frac{1}{2}(.0002909)^2 = .000000042308$. Hence, if A denote the number of degrees in an arc, and $n$ the additional minutes, then $sin\,(A + n) = 2sin\,A - 2sin\,A(n^2 \times .000,000,042308) - sin\,(A - n)$. Thus, in the equidifferent arcs of 34° 39′, 35°, and 35° 21′, the sine of the latter is equal to $1.141528\,(1 - 441 \times .000000042308) - .5685619 = .5785696$.

After the sines have been calculated up to 60°, the rest are obtained by simple addition. Thus, $sin\,61° = sin\,59° + sin\,1° = .857163 + .0174524 = .8746197$.

## TABLE OF SINES, TANGENTS and SECANTS, to every degree of the Quadrant.

| Deg. | Sine. | Cosine. | Tangent. | Cotangent. | Secant. | Cosecant. | Deg. |
|---|---|---|---|---|---|---|---|
| 1 | 0174524 | 9998477 | 0174551 | 57.289962 | 1.0001523 | 57.298688 | 89 |
| 2 | 0348995 | 9993908 | 0349208 | 28.636253 | 1.0006095 | 28.653708 | 88 |
| 3 | 0523360 | 9986295 | 0524078 | 19.081137 | 1.0013723 | 19.107323 | 87 |
| 4 | 0697565 | 9975641 | 0699268 | 14.300666 | 1.0024419 | 14.335587 | 86 |
| 5 | 0871557 | 9961947 | 0874887 | 11.430052 | 1.0038198 | 11.473713 | 85 |
| 6 | 1045285 | 9945219 | 1051042 | 9.5143645 | 1.0055083 | 9.5667722 | 84 |
| 7 | 1218693 | 9925462 | 1227846 | 8.1443494 | 1.0075098 | 8.2055090 | 83 |
| 8 | 1391731 | 9902681 | 1405408 | 7.1153697 | 1.0098276 | 7.1852965 | 82 |
| 9 | 1564345 | 9876883 | 1583844 | 6.3137515 | 1.0124651 | 6.3924532 | 81 |
| 10 | 1736482 | 9848078 | 1763270 | 5.6712818 | 1.0154266 | 5.7587705 | 80 |
| 11 | 1908090 | 9816272 | 1943803 | 5.1445540 | 1.0187167 | 5.2408431 | 79 |
| 12 | 2079117 | 9781476 | 2125566 | 4.7046301 | 1.0223406 | 4.8097343 | 78 |
| 13 | 2249511 | 9743701 | 2308682 | 4.3314759 | 1.0263041 | 4.4454115 | 77 |
| 14 | 2419219 | 9702957 | 2493280 | 4.0107809 | 1.0306136 | 4.1335655 | 76 |
| 15 | 2588190 | 9659258 | 2679492 | 3.7320508 | 1.0352762 | 3.8637033 | 75 |
| 16 | 2756374 | 9612617 | 2867454 | 3.4874144 | 1.0402994 | 3.6279553 | 74 |
| 17 | 2923717 | 9563048 | 3057307 | 3.2708526 | 1.0456918 | 3.4203036 | 73 |
| 18 | 3090170 | 9510565 | 3249197 | 3.0776835 | 1.0514622 | 3.2360680 | 72 |
| 19 | 3255682 | 9455186 | 3443276 | 2.9042109 | 1.0576207 | 3.0715535 | 71 |
| 20 | 3420201 | 9396926 | 3639702 | 2.7474774 | 1.0641778 | 2.9238044 | 70 |
| 21 | 3583679 | 9335804 | 3838640 | 2.6050891 | 1.0711450 | 2.7904281 | 69 |
| 22 | 3746066 | 9271839 | 4040262 | 2.4750869 | 1.0785347 | 2.6694672 | 68 |
| 23 | 3907311 | 9205049 | 4244748 | 2.3558524 | 1.0863604 | 2.5593047 | 67 |
| 24 | 4067366 | 9135455 | 4452287 | 2.2460368 | 1.0946363 | 2.4585933 | 66 |
| 25 | 4226183 | 9063078 | 4663077 | 2.1445069 | 1.1033779 | 2.3662016 | 65 |
| 26 | 4383711 | 8987940 | 4877326 | 2.0503038 | 1.1126019 | 2.2811720 | 64 |
| 27 | 4539905 | 8910065 | 5095254 | 1.9626105 | 1.1223262 | 2.2026893 | 63 |
| 28 | 4694716 | 8829476 | 5317094 | 1.8807265 | 1.1325701 | 2.1300545 | 62 |
| 29 | 4848096 | 8746197 | 5543091 | 1.8040478 | 1.1433541 | 2.0626653 | 61 |
| 30 | 5000000 | 8660254 | 5773503 | 1.7320508 | 1.1547005 | 2.0000000 | 60 |
| 31 | 5150381 | 8571673 | 6008606 | 1.6642795 | 1.1666334 | 1.9416040 | 59 |
| 32 | 5299193 | 8480481 | 6248694 | 1.6003345 | 1.1791784 | 1.8870799 | 58 |
| 33 | 5446390 | 8386706 | 6494076 | 1.5398619 | 1.1923633 | 1.8360785 | 57 |
| 34 | 5591929 | 8290376 | 6745085 | 1.4825610 | 1.2062179 | 1.7882916 | 56 |
| 35 | 5735764 | 8191520 | 7002075 | 1.4281480 | 1.2207746 | 1.7434468 | 55 |
| 36 | 5877853 | 8090170 | 7265425 | 1.3763819 | 1.2360680 | 1.7013016 | 54 |
| 37 | 6018150 | 7986355 | 7535541 | 1.3270448 | 1.2521357 | 1.6616401 | 53 |
| 38 | 6156615 | 7880108 | 7812856 | 1.2799416 | 1.2690182 | 1.6242692 | 52 |
| 39 | 6293204 | 7771460 | 8097840 | 1.2348972 | 1.2867596 | 1.5890157 | 51 |
| 40 | 6427876 | 7660444 | 8390996 | 1.1917536 | 1.3054073 | 1.5557238 | 50 |
| 41 | 6560590 | 7547096 | 8692867 | 1.1503684 | 1.3250130 | 1.5242531 | 49 |
| 42 | 6691306 | 7431448 | 9004040 | 1.1106125 | 1.3456327 | 1.4944765 | 48 |
| 43 | 6819984 | 7313537 | 9325151 | 1.0723687 | 1.3673275 | 1.4662792 | 47 |
| 44 | 6946584 | 7193398 | 9656888 | 1.0355303 | 1.3901686 | 1.4395565 | 46 |
| 45 | 7071068 | 7071068 | 1000000 | 1.0000000 | 1.4142136 | 1.4142136 | 45 |
| — | Cosine. | Sine. | Cotangent. | Tangent. | Cosecant. | Secant. | — |

Versed sines and supplementary versed sines are only the differences and sums of the radius and the sines.

Tangents are derived from the analogy, *cos*A : *sin*A : : R : *tan* A. Thus, *cos* 42° : *sin* 42° : : R : *tan* 42°, or .7431448 : .6691306 : : 1 : .9004040. Beyond 45°, the computation is simplified, the tangent of an arc being the reciprocal of its cotangent.

The secants are merely the reciprocals of the cosines, and may be dispensed with in practice.

From the lower tangents and secants, the tangents of arcs exceeding 45° are most easily derived, since it may be shown that $tan(A+45°)=sec\,2A+tan\,2A$. Thus, $tan\,46° =sec\,2°+tan\,2°$, or $1.0355303=1.0006095+.0349208$.

An elegant method of calculating approximate sines corresponding to any division of the quadrant, may be derived from the principles already stated: For the successive differences of the sines of the arcs A—B, A, and A+B, are $sin A—sin(A—B)$, and $sin(A+B)—sin A$; and consequently the difference between these again, or the second difference of the sines, is $sin(A + B) + sin(A — B) — 2\,sin A = —2vers B\, sin A$. The second differences of the progressive sines are hence subtractive, and proportional to the sines themselves. Wherefore the sines may be deduced from their second differences, by reversing the usual process of formation, and recompounding their separate elements. Thus, the sines of A—B and A being already known, their second and descending difference, as it is thus derived from the sine of A, will combine to form the succeeding sine of A+B, which is $— 2vers B\, sin A + (sin A — sin (A — B)) + sin A$. It only remains then, to determine, in any trigonometrical system, the constant multiplier of the sine, or twice the versed sine of the component arc.

This very simple computation is admirably adapted to

the decimal scale of numeration, and the nautical division of the circle. Suppose a quadrant to contain 16 equal parts, or *half points of the mariner's compass*; the length of each arc, the radius being unit, is nearly $\frac{22}{7}.\frac{1}{32}=\frac{11}{112}$, and consequently twice its versed sine is $(\frac{11}{112})^2$, or, in round numbers, $\frac{1}{103}$. It will be sufficiently accurate, therefore, to employ 100 for the constant divisor. The sine of the first arc or half point being likewise expressed by 100, let the nearer integral quotients be always retained, and the sine of the whole quadrant, or the radius itself, will come out exactly 1000. The first term being divided by 100 gives 1 for the second difference, which subtracted from 100, leaves 99 for the first difference, and this joined to 100, forms the second term. Again, dividing 199 by 100, the quotient 2 is the second difference, which, taken from 99, leaves 97 for the first difference, and this added to 199, gives the third term. In like manner, the rest of the terms are found, as in the following table.

| Half points. | Arcs. | Sines. | 1st Diff. | 2d Diff. | Excess. | Correct Sines. |
|---|---|---|---|---|---|---|
| 1 | 5° 37½′ | 100 | 99 | 1 | 3 | 97 |
| 2 | 11 15 | 199 | 97 | 2 | 4 | 195 |
| 3 | 16 52½ | 296 | 94 | 3 | 5 | 291 |
| 4 | 22 30 | 390 | 90 | 4 | 6 | 384 |
| 5 | 28 7½ | 480 | 85 | 5 | 7 | 473 |
| 6 | 33 45 | 565 | 79 | 6 | 8 | 557 |
| 7 | 39 22½ | 644 | 73 | 6 | 9 | 635 |
| 8 | 45 00 | 717 | 66 | 7 | 10 | 707 |
| 9 | 50 37½ | 783 | 58 | 8 | 9 | 774 |
| 10 | 56 15 | 841 | 50 | 8 | 8 | 833 |
| 11 | 61 52½ | 891 | 41 | 9 | 7 | 884 |
| 12 | 67 30 | 932 | 32 | 9 | 6 | 926 |
| 13 | 73 7½ | 964 | 22 | 10 | 5 | 959 |
| 14 | 78 45 | 986 | 12 | 10 | 4 | 982 |
| 15 | 84 22½ | 998 | 2 | 10 | 3 | 995 |
| 16 | 90 00 | 1000 | | | | |

The errors occasioned by neglecting the fractions accumulate at first, but afterwards gradually diminish, from the effect of compensation. The greatest deviation takes place, as might be expected, at the middle arc, whose sine is 707 instead of 717. Reckoning the error in excess as limited by 10, and declining uniformly on each side, the correct sines are finally deduced. The numbers thus obtained seldom differ, by the thousandth part, from the truth, and are hence far more accurate than the practice of navigation ever requires.

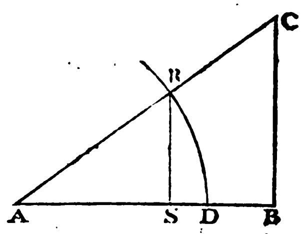
This simple and expeditious mode of forming the sines is not merely an object of curiosity, but may be deemed of very considerable importance, since it will enable the mariner, altogether independent of the aid of books, to the loss of which he is often exposed by the hazards of the sea, to construct a table of *departure* and *difference of latitude*, sufficiently accurate for every real purpose.

## PROP. I. THEOR.

**In a right angled triangle, the radius is to the sine of an oblique angle, as the hypotenuse to the opposite side.**

Let the triangle ABC be right angled at B; then *R* : *sin*CAB : : AC : CB.

For, in the base AB, assume AR equal to the given or tabular radius, describe the arc RD, and let fall the perpendicular RS. The triangles ARS and ACB are evidently similar, and therefore AR : RS : : AC : CB. But, AR being the radius,




RS is the sine of the arc RD which measures the angle RAD or CAB; and consequently *R* : *sin*A : : AC : CB.

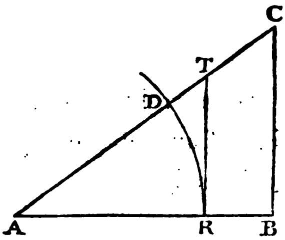
*Cor.* Hence the radius is to the cosine of an angle, as the hypotenuse to the adjacent side; for R : *sin*C or *cos*A : : AC : AB.

## PROP. II. THEOR.

**In a right angled triangle, the radius is to the tangent of an oblique angle, as the adjacent side to the opposite side.**

Let the triangle ABC be right angled at B; then *R* : *tan*BAC : : AB : BC.

For, assuming AR as before equal to the given radius, describe the arc RD, and erect the perpendicular RT. The triangles ART and ABC being similar, AR : RT : : AB : BC. But, AR being the radius, RT is the tangent of the arc RD which measures the angle at A; and therefore *R* : *tan*A : : AB : BC.



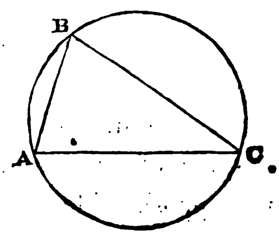

*Cor.* Hence the radius is to the secant of an angle, as the adjacent side to the hypotenuse. For AT is the secant of the arc RD, or of the angle at A; and, from similar triangles, AR : AT : : AB : AC.

## PROP. III. THEOR.

The sides of any triangle are as the sines of their opposite angles.

In the triangle ABC, the side AB is to BC, as the sine of the angle at C to the sine of that at A.

For let a circle be described about the triangle; and the sides AB and BC, being chords of the intercepted arcs or of the angles at the centre, are equal to twice the sines of the halves of those angles, or the angles ACB and CAB at the circumference. But, of the same angles, the chords or sines are proportional to the radius; and consequently AB : BC : : *sin*C : *sin*A.



*Cor.* Since the straight lines AB and BC are chords, not only of the arcs AB and BC, but of the arcs ACB and BAC, or the defects of the former from the circumference, it follows that the sides of the triangle are proportional likewise to the sines of half these compound arcs, or to the sines of the supplements of their opposite angles.

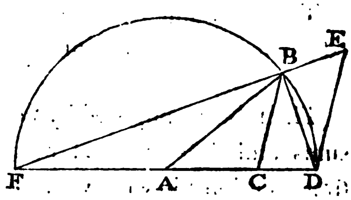
## PROP. IV. THEOR.

In any triangle, the sum of two sides is to the difference, as the tangent of half the sum of the angles at the base, to the tangent of half their difference.

In the triangle ABC,

$$AB+AC : AB-AC : : tan\frac{C+B}{2} : tan\frac{C-B}{2}.$$

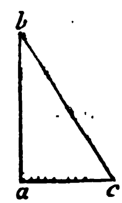
From the vertex A, with a distance equal to the greater side AB, describe the semicircle FBD, meeting the other side AC extended both ways to F and D, join BD and BF, and produce the latter to meet a straight line DE drawn parallel to CB.




Because the isosceles triangle DAB, has the same vertical angle with the proposed triangle CAB, each of its remaining angles ADB and ABD is equal to half the sum of the angles ACB and ABC; and therefore the defect of ABC from that mean, that is the angle CBD, or its alternate angle BDE, must

be equal to half the difference of those angles. Now FBD being a right angle, BF and BE are tangents of the angles BDF and BDE, to the radius DB, and hence are proportional to the tangents of those angles with any other radius. But since CB and DE are parallel, CF or AB+AC : CD or AB—AC : : BF : BE; consequently $AB+AC : AB-AC :: tan\frac{ACB+ACB}{2} : tan\frac{ACB-ABC}{2}$, or $AB+AC : AB-AC :: cot\,\frac{1}{2}A : cot\,(B+\frac{1}{2}A)$, or $-cot\,(C+\frac{1}{2}A.)$

*Cor.* Suppose another triangle *abc* to have the sides *ab* and *ac* equal to AB and AC, but containing a right angle: It is obvious that $tan\frac{c+b}{2} : tan\frac{c-b}{2}$ $:: tan\frac{ACB+ABC}{2} : tan\frac{ACB-ABC}{2}$, or




$R : tan(45°-b) :: tan\frac{ACB+ABC}{2} : tan\frac{ACB-ABC}{2}$, that is,

$R : tan(45-b) :: cot\frac{1}{2}A : cot(B+\frac{1}{2}A)$, or $-cot(C+\frac{1}{2}A)$. Now, in the right angled triangle *abc*, the base *ab* or AB, is to the perpendicular *ac*, or AC, as the radius, to the tangent of the angle at *b*.

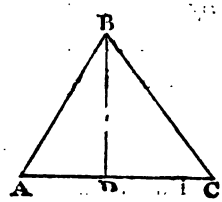
This solution, by a double analogy, is the best adapted for logarithmic calculation.

The tables computed by Gauss, of the logarithms of the sums and differences of numbers, render, however, the first solution likewise very convenient.

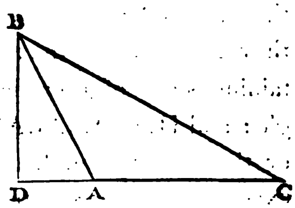
## PROP. V. THEOR.

**In any triangle, as twice the rectangle under two sides, is to the difference between their squares and the square of the base, so is the radius to the cosine of the contained angle.**

In the triangle ABC, $2AB.AC : AB^2+AC^2-BC^2 : : R : cos BAC$; the angle BAC being acute or obtuse, according as $BC^2$ is less or greater than $AB^2+AC^2$.




For let fall the perpendicular BD. In the right angled triangle ADB, $AB : AD : : R : sin ABD$ or $cos BAC$; consequently, by taking like multiples, $2AB.AC : 2AD.AC : : R : cos BAC$. But twice the rectangle under AD and AC is equal to the difference of the squares AB and AC from the square of BC. Whence $2AB.AC : AB^2+AC^2-BC^2 : : R : cos BAC$.




*Cor.* The radius being denoted by unit, it follows that $AB^2+AC^2-BC^2=2AB.AC.cos BAC$, and consequently $BC^2 = AB^2 + AC^2 - 2AB.AC\ cos BAC$, or $BC = \sqrt{(AB^2+AC^2-2AB.AC\ cos BAC)}$.

## PROP. VI. THEOR.

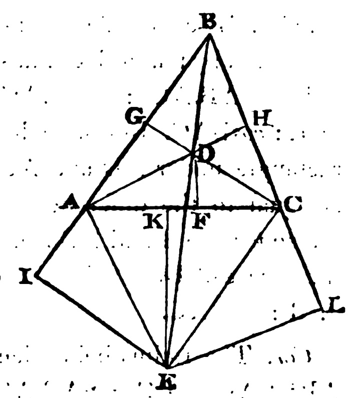
In any triangle, as the rectangle under two sides is to the rectangle under the excesses of the semiperimeter above those sides, so is the square of the radius, to the square of the sine of half their contained angle.

In the triangle ABC, the perimeter being still denoted by P, AB.BC : ($\frac{1}{2}$P—AB) ($\frac{1}{2}$P—BC) : : $R^2$ : $sin\frac{1}{2}B^2$.

For, the same construction being retained, in the right-angled triangles BIE and BGD, BE : IE : : R : $sin\frac{1}{2}$B,

and BD : GD : : R : $sin\frac{1}{2}$B ;

whence BE.BD : IE.GD : : $R^2$ : $sin\frac{1}{2}B^2$.




But it has been proved that BE.BD=AB.BC, or the rectangle under the containing sides of the triangle; and IE.GD=AI.AG, or the rectangle under the excesses of the semiperimeter above the sides AB and BC. Wherefore the proposition is established.

This proposition might be superseded in practice by the preceding; but it is more compact, and better fitted for the calculation by logarithms.

The six preceding theorems contain the elements of trigonometrical calculation. A triangle has only five variable parts—the three sides and two angles, the remaining angle being merely supplemental. Now, it is a general principle, that, three of those parts being given, the rest may be thence determined. But the right-angled triangle has necessarily one known angle; and, consequently, the opposite side is deducible from the containing sides. In right-angled triangles, therefore, the number of parts is reduced to four, any two of which being the assigned, the others may be found.

## PROP. VII. PROB.

Two variable parts of a right-angled triangle being given, to find the rest.

This problem divides itself into four distinct cases, according to the different combination of the data.

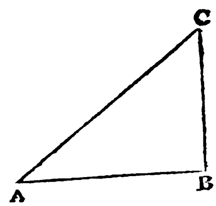
1. *When the hypotenuse and a side are given.*

2. *When the two sides containing the right angle are given.*

3. *When the hypotenuse and an angle are given.*

4. *When either of the sides and an angle are given.*

The first and third cases are solved by the application of Proposition I. and the second and fourth cases receive their solution from Proposition II. It may be proper, however, to exhibit the several analogies in a tabular form.




| Case. | Given. | Sought. | SOLUTION. |
|---|---|---|---|
| I. | AC, AB, | A, or C, BC, | AC : AB : : R : *sin*C, or *cos*A. R : *sin*A : : AC : BC. |
| II. | AB, BC, | A, or C, AC, | AB : BC : : R : *tan*A, or *cot* C. *cos*A : R : : AB : AC. |
| III. | AC, A, | AB, BC, | R : *cos*A : : AC : AB. R : *sin*A : : AC : BC. |
| IV. | AB, A, | BC, AC, | R : *tan*A : : AB : BC. *cos*A : R : : AB : AC. |

In the first and second cases, BC or AC might also be deduced from Elementary Geometry.

For $AC^2 = AB^2 + BC^2$, or $AC = \sqrt{(AB^2+BC^2)}$,
and $BC^2 = AC^2 - AB^2 = (AC+AB)\ (AC-AB)$,
or $BC = \sqrt{(AC+AB)\ (AC-AB)}$.

*Cor.* Hence the first case admits of a simple approximation. For, it appears, that, AC being made the radius, 2AC+AB is to 3AC, as the side BC is to the arc which measures its opposite angle CAB, or alternately 2AC+AC is to BC, as 3AC to the arc corresponding to BC. But it may be shewn that the radius is equal to an arc of $57^\circ\ 17'\ 44''\ 48'''$, or $57\frac{1}{3}^\circ$ nearly; wherefore 3AC is to the arc

which corresponds to BC, as $3 \times 57\frac{1}{3}^{\circ}$, or 172°, to the number of degrees contained in the angle CAB, and consequently $2AC+AB : BC :: 172^{\circ}$ : the expression of the angle at A, or $AC+\frac{1}{2}AB : BC :: 86^{\circ}$ : number of degrees in the angle at A.

This approximation will be the more correct, when the side opposite to the required angle becomes small in comparison with the hypotenuse; but the quantity of error in any case can never amount to 4 minutes.

## PROP. VIII. PROB.

Three variable parts of an oblique angled triangle being given, to find the other two.

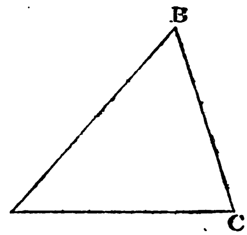
This general problem includes three distinct cases, one of which again is branched into two subordinate divisions.

1. *When all the three sides are given.*

2. *When two sides and an angle are given; which angle may either* (1.) *be contained by these sides, or* (2.) *subtended by one of them.*

3. *When a side and two of the angles are given.*

The first case receives two different solutions, derived from Propositions V. and VI. which have their respective advantages. The second case, consisting of two branches, is resolved by the application of Propositions III. and IV.; and the resolution of the third case flows immediately from the former of these propositions.




| Case. | | Given. | Sought. | SOLUTION. | |
|---|---|---|---|---|---|
| I. | | AB, BC, and AC. | B. | $AB \cdot BC : (\frac{1}{2}P-AB)(\frac{1}{2}P-BC) :: R^2 : \sin\frac{1}{2}B^2$ | 1 |
| | | | | $\frac{1}{2}P(\frac{1}{2}P-AC) : (\frac{1}{2}P-AB)(\frac{1}{2}P-BC) :: R^2 : \tan\frac{1}{2}B^2$ | 2 |
| | | | | $AB \cdot BC : \frac{1}{2}P(\frac{1}{2}P-AC) :: R^2 : \cos\frac{1}{2}B^2$ | 3 |
| | | | | $2AB \cdot BC : AB^2 + BC^2 - AC^2 :: R : \cos B.$ | 4 |
| II. | 1 | AB, BC, and C. | A. and AC. | $AB : BC :: \sin C : \sin A$; whence B, and | 5 |
| | | | | $\sin C : \sin B :: AB : AC$ | 6 |
| | 2 | AB, BC, and B. | A, or C, and AC. | $AB+BC : AB-BC :: \cot\frac{1}{2}B : \cot(A+\frac{1}{2}B)$, or $-\cot(C+\frac{1}{2}B)$. | 7 |
| | | | | $\begin{cases} AB : BC :: R : \tan b; \text{ and} \\ R : \tan(45^\circ-b) :: \cot\frac{1}{2}B : \cot(A+\frac{1}{2}B), \text{ or } -\cot(C+\frac{1}{2}B). \end{cases}$ | 8 |
| | | | | $\sin A : \sin B :: BC : AC$, or | 9 |
| | | | | $AC = \sqrt{(AB^2+BC^2-2AB.BC\cos B.)}$ | 10 |
| III. | | AB, A, B, and thence C. | BC, AC. | $\sin C : \sin A :: AB : BC.$ | 11 |
| | | | | $\sin C : \sin B :: AB : AC.$ | 12 |

For the resolution of the first Case, the analogy set down first, is on the whole the most convenient, particularly if the angle sought should not be very obtuse. The second analogy may be applied with obvious advantage through the entire extent of angles. The third and fourth

analogies, especially the latter, are not adapted for the calculation of very acute angles; they will, however, answer the best when the angle sought is obtuse. It is to be observed, that the cosines of an angle and of its supplement are the same, only placed in opposite directions; and hence the second term of the analogy, or the difference of $AB^2+BC^2$ from $AC^2$, is in excess or defect, according as the angle at B is acute or obtuse. These remarks are founded on the unequal variation of the sine and tangent, corresponding to the uniform increase of an arc.

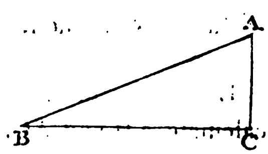
The first part of Case II. is ambiguous, for an arc and its supplement have the same sine. This ambiguity, however, is removed if the character of the triangle, as acute or obtuse, be previously known.

For the solution of the second part of Case II. the first analogy is the most usual, but the double analogy is the best adapted for logarithms. In astronomy, this mode of calculation is particularly commodious. The direct expression for the side subtending the given angle is very convenient, where logarithms are not employed.

## PROP. IX. PROB.

Given the horizontal distance of an object and its angle of elevation, to find its height and absolute distance.

Let the angle ABC, which an object A makes at the station B with an horizontal line, and also the distance BC of the perpendicular AC, to find that perpendicular, and the hypotenusal or aërial distance BA.




In the right angled triangle BCA, the radius is to the tangent of the angle at B, as BC to AC; and the radius is

to the secant of the angle at B, or the cosine of the angle at B is to the radius, as BC to AB.

## PROP. X. PROB.

Given the acclivity of a line, to find its corresponding vertical and horizontal lengths.

In the preceding figure, the angle CBA and the hypotenusal distance BA being given, to find the height AC and the horizontal distance BC of the extremity A.

The triangle BCA being right angled, the radius is to the sine of the angle CBA as BA to AC, and the radius is to the cosine of CBA as BA to BC.

If the acclivity be small, and A denote the measure of that angle in minutes; then $AC = BA \times \frac{A}{3438}$ nearly.

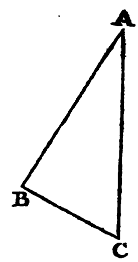
In most cases when CAB is a small angle, the horizontal distance may be computed with sufficient exactness, by deducting $\frac{AC^2}{2BA}$ from the hypotenusal distance.

## PROP. XI. THEOR.

Given the interval between two stations, and the direction of an object viewed from them, to find its distance from each.

Let BC be given, with the angles ABC and ACB, to calculate AB and AC.

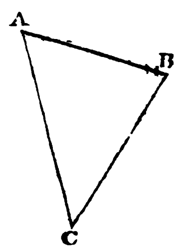
In the triangle CBA, the angles ABC and ACB being given, the remaining or supplemental angle BAC is thence given; and consequently, $\sin BAC : \sin ACB :: BC : AB$, and $\sin BAC : \sin ABC :: BC : AC$.




## PROP. XII. PROB.

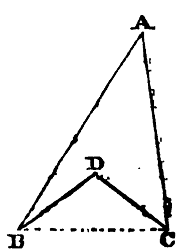
**Given the distances of two objects from any station and the angle which they subtend, to find their mutual distance.**

Let AC, BC, and the angle ACB be given, to determine AB.




In the triangle ABC, since two sides and their contained angle are given, therefore $AC + BC : AC - BC :: \cot\frac{1}{2}C : \cot(A+\frac{1}{2}C)$, then $\sin A : \sin C :: BC : AB$; or $AB = \sqrt{(AC^2 + BC^2 - 2AC.BC.\cos C.)}$

*Cor.* By combining this with the preceding proposition, the distance of an object may be found from two stations, between which the communication is interrupted. Thus, let A be visible from B and C, though the straight line BC cannot be traced. Assume a third station D, from which B and C are both seen. Measure DB and DC, and observe the angles BDC, ABC and ACB. In the triangle BDC, the base BC is



found as above; and thence, by the preceding proposition, the sides AB and AC of the triangle ABC are determined.

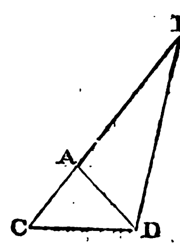
## PROP. XIII. PROB.

Given the interval between two stations, and the directions of two remote objects viewed from them in the same plane, to find the mutual distance, and relative position of those objects.

Let the points A, B represent the two objects, and C, D the two stations from which these are observed; the interval or base CD being measured, and also the angles CDA, CDB at the first station, and DCA, DCB at the second; it is thence required to determine the transverse distance AB, and its direction.

It is obvious that each of the points A and B would be assigned geometrically by the intersection of two straight lines, and consequently that the position of the objects will not be determined, unless each of them appears in a different direction at the successive stations.

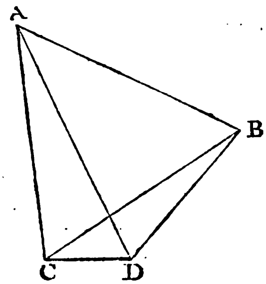
1. *Suppose one of the stations C to lie in the direction of the two objects A and B.*




At C observe the angle BCD, and at D the angles CDA and BDC. Then by Prop. IX. *sin*CAD : *sin*CDA : : CD : CA, and *sin*CBD : *sin*CDB : : CD : CB; the difference or sum of CA and CB is AB, the distance sought.

*2. When neither station lies in the direction of the two objects, and the base CD has a transverse position.*

Find the distances AC and BC of both objects from one of the stations C; then, the contained angle ACB, or the excess of DCA above DCB, being likewise given, the angles at the base AB of the triangle BCA, and the base itself, may be calculated. For AC+BC : AC—BC : : *cot*$\frac{1}{2}$ACB : *cot*($\frac{1}{2}$ACB+CAB),




and thus the angle CAB is found. Or more conveniently by two successive operations, AC : BC : : *R* : *tan b*, and R : *tan*(45°—*b*) : : *cot*$\frac{1}{2}$ACB : *cot*($\frac{1}{2}$ACB+CAB.) Now, *sin*CAB : *sin*ACB : : BC : AB, or AB$=\sqrt{}$(AC$^2$+BC$^2$ —2AC.BC *cos*ACB).

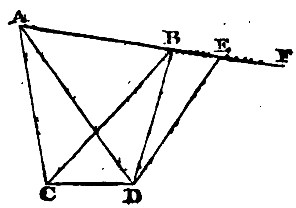
The inclination of AB to CD in the first case is given by observation, and in the second case it is evidently the supplement of the interior angles CAB and DCA. A parallel to AB may hence be drawn from either station.

*Cor.* Hence the converse of this problem is readily solved. Suppose two remote objects A and B, of which the mutual distance is already known, are observed from the stations C and D, and it were thence required to determine the interval CD. Assume unit to denote CD, and calculate AB according to the same scale of measures; the actual distance AB being then divided by that result, will give CD: For the several triangles which combine to form the quadrilateral figure CABD, are evidently given in species.

## PROP. XIV. PROB.

**Given the directions of two inaccessible objects, viewed in the same plane from two given stations, to trace the extension of the straight line connecting them.**

Let the angles ACD, BCD be observed at C, and ADC, BDC at D, with the base CD; to find a point E in the straight line ABF produced through A and B.




By the last proposition, find AD and the angle DAB, and assume any angle ADE. In the triangle DAE, the angles at the base AD, and consequently the vertical angle AED, being known, it follows, that *sin*AED : *sin*EAD : : AD : DE. Wherefore, measure out DE on the ground, and its extremity E will mark the extension of AB.

## PROP. XV. PROB.

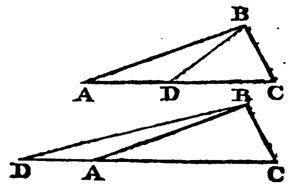
**The mutual distances of three remote objects being given, with the angles which they subtend at a station in the same plane, to find the relative place of that station.**

Let the three points A, B, and C, and the angles ADB

and BDC which they form at a fourth point D, be given; to determine the position of that point.

1. *Suppose the station D to be situate in the direction of two of the objects A, C.*

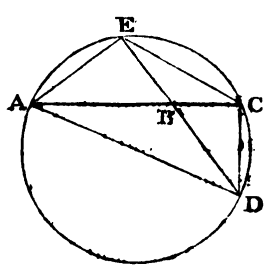
All the sides AB, AC and BC of the triangle ABC being given, the angle BAC is found by Case I.; and in the triangle ABD, the side AB with the angles at A and D being given, the side AD is found by Case III., and consequently the position of the point D is determined.




2. *Suppose the three objects A, B and C to lie in the same direction.*

Describe a circle about the extreme objects A, C and the station D, join DA, DB and DC, produce DB to meet the circumference in E, and join AE and CE.

In the triangle AEC, the side AC is given, and the angles EAC and ECA, being equal to CDE and ADE, are consequently given; wherefore the side AE is found by Case III. The triangle AEB, having thus the sides AE, AB, and their contained angle EAB or BDC given, the angle ABE, and its supplement ABD are found by Case II. Lastly, in the triangle ABD, the angles ABD and ADB, with the side AB, are given; whence BD is found by Case III. But since the angle ABD and the distance DB are assigned, the position of the station D is evidently determined.

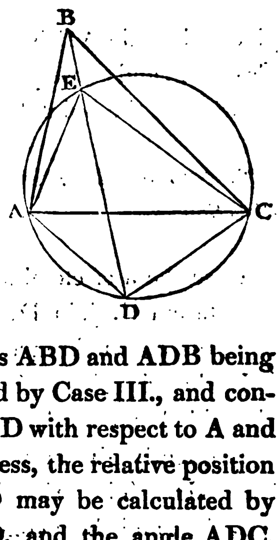


3. *Let the three objects form a triangle, and the station D lie either without or within it.*

Through D and the extreme points A and C describe a circle, draw DB cutting the circumference in E, and join AE and CE.

1. In the triangle AEC, the side AC, and the angles ACE and CAE, which are equal to ADB and BDC, being given, the side AE is found by Case III.

2. All the sides of the triangle ABC being given, the angle CAB is found by Case I.




3. In the triangle BAE, the sides AB and AE are given, and their contained angle EAB, or the difference of CAE and CAB, are given, whence, by Case II., the angle ABE or ABD is found.

4. Lastly, in the triangle DAB, the side AB and the angles ABD and ADB being given, the side AD or BD is found by Case III., and consequently the position of the point D with respect to A and B is determined. By a like process, the relative position of D and C is deduced; or CD may be calculated by Case II. from the sides AC, AD, and the angle ADC, are given in the triangle CAD.

It is obvious that the calculation will fail, if the points B and E should happen to coincide. In fact, the circle then passing through B, any point D whatever in the opposite arc ADC will answer the conditions required, since the angle ADB and BDC, being now in the same segment, must remain unaltered.

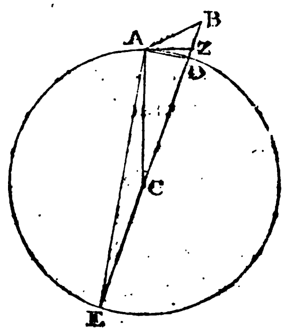
In all the foregoing problems, the angles on the ground are supposed to be taken by means of a *theodolite*; which, being adjusted by means of spirit-levels, measures only horizontal and vertical angles, or decomposes other angles into these elements. If the *sextant* or the *repeating circle* be employed for the same purpose, such angles, when oblique, must be reduced by calculation to their projections on the horizontal plane.

In surveying an extensive country, a base is first carefully measured; and the directions of the prominent distant objects being observed from both of its extremities, they are all connected with it by a series of triangles. To avoid, in practice, the multiplication of errors, these triangles should be chosen, as nearly as possible, equilateral. After a similar method, large estates are the most correctly planned and measured; the ordinary practice of carrying the theodolite with a chain merely round the extreme boundary being subject to much inaccuracy.

If the inequality of the surface of the ground will not admit of the measurement of a base of a sufficient length, a smaller one may be selected at first, and another base derived from this, by combining with it one or more triangles. These triangles, to preclude the multiplication of errors, should be as nearly as possible right-angled, and similar, having their sides increasing in a continued proportion. When this rate of increase is not less than the ratio of the radius to the side of an inscribed equilateral triangle, the number of intermediate triangles between the measured and the computed base may be shown to be rather favourable, on the whole, to the accuracy of the result.

The vertical angles employed in the mensuration of

heights, since they are estimated from the varying direction of the level or the plummet, must evidently, when the stations are distant, require some correction. Let the points A and B represent two remote objects, and C their centre of gravitation; with the radius CA describe a circle, draw CB cutting the circumference in D and E, and join EA and AD. The converging lines AC and BC will indicate the direction of the plummet at A and B, the intercepted arc AD will trace the contour of a quiescent fluid, and the tangent AZ, being applied to A, will mark the line of the horizon seen from that station. Wherefore the vertical angle of the remote object B, as observed at A, is only ZAB, which is less than the true angle DAB, by the exterior angle DAZ. But DAZ being equal to the angle AED in the alternate segment, is equal to half the angle ACD at the centre. Hence the true vertical angle at any station will be found, by adding to the observed angle half the measure of the intercepted arc; and this measure depending on the curvature of the earth, which is neither uniform nor quite regular, must be deduced, for each particular place, from the length of the corresponding degree of latitude.




Such nicety, however, is very seldom required. It will be sufficiently accurate in practice to assume the mean quantities, and to consider the earth as a globe, whose circumference is 24,856 English miles, and its diameter 7,912. The arc of a minute on the meridian being, therefore, equal to 6076 feet, the correction to be added to the observed

vertical angle must amount to one second, for every 69 yards contained in the intervening distance.

The quantity of depression ZD below the horizon is hence easily computed; for $AZ^2 = EZ.ZD$, or very nearly ED.ZD; and, consequently, since the diameter ED is constant, the visual depression of an object is proportional to the square of its distance AZ from the observer. In the space of one mile, this depression will amount to $\frac{5280}{7912}$ parts of a foot; and generally, therefore, it may be expressed in feet, by two-thirds of the square of the distance in miles. Thus, at the distance of twenty miles, the depression is $266\frac{2}{3}$ feet; and at that of fifty miles, it amounts to $1666\frac{2}{3}$, or nearly the third of a mile.

But the effect of the earth's curvature is modified by another cause, arising from optical deception. An object is never seen by us in its true position, but in the direction of the ray of light which conveys the impression. Now the luminous particles, in traversing the atmosphere, are, by the force of superior attraction, refracted or bent continually towards the perpendicular, as they penetrate the lower and denser layers; and consequently they describe a curved track, of which the last portion, or its tangent, indicates the apparent elevation of a remote point. This trajectory, suffering almost a regular inflextion, may be considered as very nearly an arc of a circle, which has for its radius six times the radius of our globe. Hence, to correct the error occasioned by refraction, it will only be requisite to diminish the effects of the earth's curvature by one-sixth part, or to deduct, from the vertical angles, the twelfth part of the measure of the intervening terrestrial arc. The quantity of horizontal refraction, however, as it depends on the density of the air at the surface, is extremely variable, especially in our unsteady climate.

# NOTES AND ILLUSTRATIONS.

## INTRODUCTION.

THE doctrine of proportion has given rise to much controversy. In their mode of treating that important subject, Mathematicians have differed very widely; some rejecting the procedure of Euclid as circuitous and embarrassed, while others appear disposed to extol it as one of the happiest and most elaborate monuments of human ingenuity. But, to view the matter in its true light, we should endeavour previously to dispel that mist which has so long obscured our vision. The Fifth Book of Euclid, in its original form, is not found to answer the purpose of actual instruction; and this remarkable and indisputed fact might alone excite suspicion of its intrinsic excellence. The great object which the framer of the Elements had proposed to himself, by adopting such an artificial definition of proportion, was to obviate the difficulties arising from the consideration of incommensurable quantities. Under the shelter of a certain indefinitude of principle, he has contrived rather to evade those difficulties than fairly to meet them. Through the whole contexture of the Elements, we may discern the influence of that mysticism which prevailed in the Platonic school; and the language used in the Fifth Book might often seem to imply, that *Ratios* are not mere conceptions of the mind, but have a real and substantial essence.

The notion of proportionality necessarily involves the idea of *number*; and the doctrine of proportion hence constitutes a branch of universal arithmetic. The properties themselves are extremely

simple, and may be regarded as only the exposition of the same principle under different aspects. The various transformations of analogies resemble exactly the changes effected in the reduction of equations in Algebra.

It is very unfortunate that, from the poverty of language, and the slow progress of science, the terms used in common life, though unavoidably deficient in precision, were adopted into Geometry. But the vagueness of expression is nowhere more apparent than in the doctrine of Proportion.—Thus, the words denoting *time* are, in most dialects, blended with those which signify *number*. To express how often a part is contained in a whole, we intimate how many *ways* it is to be placed, how many *foldings* are required, or how many *times* the operation of admeasurement must be repeated. In the Greek and Latin languages, the adverbs compounded from *plica*, a *fold*, are very extensive. In English, the corresponding terms are limited, and obviously mark their composition: for *duplex*, *triplex*, *quadruplex*, we have *double*, *triple* or *quadruple*, *twofold*, *threefold*, or *fourfold*. But our application of the word *way* is still more confined: we have only *twice* and *thrice*, or *two ways* and *three ways*. When we seek to go farther, we are absolutely obliged to borrow the word *time*; thus, we say that one number is four or five *times* greater than another; or that it would require the addition of the part so often, to form the whole.

In Proposition fifteenth, the numerical expression of the ratio A : B, may be deduced indirectly, from the series of quotients obtained in the operation for discovering their common measure.

Let A contain B, $m$ times, with a remainder C; B contain C, $n$ times, with a remainder D; and, lastly, suppose C to contain D, $p$ times, with a remainder E, and which is contained in D, $q$ time, exactly. Then $D=qE$, $C=pD+E$, $B=nC+D$, and $A=mB+C$, whence the terms D, C, B, and A, are successively computed, as multiples of E; A and B will therefore be found to contain E their common measure K and L times, or the numerical expression for the ratio of those quantities is K : L.

But it is more convenient to derive the numerical ratio from the quotients of subdivision in their natural order; and this method has besides the peculiar advantage of exhibiting a succession of elegant approximations.

Nor is it requisite, in finding the ratio of A to B, to know the values of the successive remainders C, D, E, &c. Suppose the subdivision to terminate at B; then $A=mB$, and consequently $A : B :: m : 1$. If the subdivision extend to C, then $A\pm m'C$, and $B=nC$; whence $A : B :: m' : n$. In general, therefore, the second term, in the expressions for A and B, may be rejected, and the letter which precedes it considered as the ultimate measure, corresponding to the arithmetical unit. Hence, resuming the substitutions, it follows, that the ratio of A to B may thus be successively represented:

1. $m : 1$.
2. $mn+1 : n$, or $m' : n$.
3. $m'p+m : np+11$, or $m'' : n'$.
4. $m''q+m' : n'q+n$, or $m''' : n''$.

&c. &c. &c.

The formation of these numbers will evidently stop, when the corresponding subdivision terminates. But even though the successive decomposition should never terminate, as in the case of incommensurable quantities,—yet the expressions thus obtained must constantly approach to the ratio of A : B, since they suppose only the omission of the remainder of the last division, which is perpetually diminishing.

It may be worth while to select a remarkable exemplification. Let it be required to express approximately the fractional portion of 24 hours, by which the solar year exceeds 365 days. This excess, or 5 hours, 48 minutes, and 50 seconds, being reduced to seconds, makes 20930, while 24 hours give 86400. But the successive quotients corresponding to the ratio of 86400 to 20930 are 4, 7, 1, 4, and 4, without pushing the last division with rigour. Hence the approximate ratios are 4 : 1, 29 : 7, 33 : 8, 161 : 39, and 677 : 164.

Some of these numbers deserve attention. The ratio of 4 to 1 indicates the insertion of a day every four years, being the correc-

tion of the Kalendar by the *Bissextile* or *Leap year* introduced by Julius Cæsar. The ratio of 33 to 8 indicates a more correct intercalation of 8 days in 33 years, a method adopted about six centuries ago by the Persian astronomers, who, after the lapse of seven ordinary leap years, always deferred the eighth return of the period one year later than usual. If to the products of 33 and 8 by 12, the terms 4 and 1 were added, the ratio of 400 to 97 would be found and nearly the same. This last represents the intercalation established by Pope Gregory XIII. in 1582. It implies the insertion of 97 days in the space of 400 years; which is performed by combining the Julian system with an omission of three intercalary days in four centuries; that is, the last year of each century, which happens to be a leap year, is not considered as such, unless the number of the century itself be divisible by four.

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## BOOK I.

### DEFINITIONS.

THE term *Surface* conveys a very just idea, marking the abstract external aspect, or the mere expansion which a body presents to our sense of sight. *Line* originally signified a *stroke;* and, in reference to the operation of writing, it expresses the boundary or contour of a figure. A straight line has two radical properties which are distinctly marked in different languages. It holds the same undeviating course—and it traces the shortest distance between its extreme points.

The word *Point* in every language means a *mark*, thus indicating its essential character, of denoting position.

The neatest and most comprehensive description of a *point* was given by Pythagoras, who defined it to be "an atom having position." Plato represents the constitution of a point, as *adamantine;*

alluding to the notion which then prevailed, that the diamond is absolutely indivisible.

The just conception of an *Angle* is one of the most difficult in Elementary Geometry. The term corresponds, in most languages, to *corner*, and therefore exhibits a most imperfect picture of the object designed. The attempts of the ancients to define it were most vague and defective. Before Trigonometry was cultivated, no just idea indeed could be formed of angular magnitude.

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## BOOK II.

### PROPOSITIONS.

Proposition thirteenth. This proposition is of great use in practical geometry, since it enables us to divide a triangle, of which all the sides are given, into two right-angled triangles, by determining the position, and consequently the length, of the perpendicular.

Thus, suppose the base of the triangle to be 15, and the two sides 13 and 14: Then $15^2+13^2-14^2=225+169-196=198$, which shows that the perpendicular falls within the triangle; and $\frac{198}{30}=6.6$, the segment adjacent to the short side, whence the perpendicular$=\sqrt{((13)^2-(6.6)^2)}=\sqrt{(169-43.56)}=11.2$. The area is therefore $15\times5.6=84$.

Again, if the base were 10, and the sides 21 and 17: Then $21^2-17^2-10^2=441-289-100=52$, which indicates the perpendicular to fall beyond the base. Whence $\frac{52}{20}=2.6$, the external segment; and $\sqrt{(17^2-2.6^2)}=\sqrt{(289-6.76)}=\sqrt{282.24}=16.8$, which gives 84 for the area, the same as before, and a very remarkable coincidence of two triangles.

Lastly, let the base be 9, and the two sides 17 and 10: Then $17^2-9^2-10^2=289-81-100=108$, indicating that the perpen-

dicular falls without the base. Wherefore, $\frac{108}{18}=6$, the external segment, and $\sqrt{(10^2-6^2)}=\sqrt{(100-36)}=\sqrt{64}=8$, the perpendicular; which gives $\frac{9\times 8}{2}=36$, for the area of the triangle.

The same results are obtained by applying the twenty-first proposition. Thus, in the first example, the distance of the perpendicular from the middle of the base is $\frac{14^2-13^2}{30}=.9$, and therefore the segments of the base are 8.4, and 6.6. In the second example, the distance of the perpendicular from the middle of the base is $\frac{17^2-10^2}{18}=10.5$, and consequently the segments of the base are 15 and 6. In the last example, the distance of the perpendicular from the middle part of the base is $\frac{21^2-17^2}{20}=76$, and the segments of that base are hence 12.6 and 2.6. The length of the perpendicular and the area of the triangle are, in each case, therefore, easily deduced from these data.

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Since rectangles correspond to numerical products, the properties of the sections of lines are easily derived from symbolical arithmetic or algebra.

1. In Prop. 5. let the two lines be denoted by $a$ and $b$; then $(a+b)^2=a^2+b^2+2ab$.

2. In Prop. 6. let the two lines be denoted by $a$ and $b$; then $(a-b)^2=a^2+b^2-2ab$.

3. In Prop. 7. let the two lines be denoted by $a$ and $b$; then $(a+b)(a-b)=a^2-b^2$.

4. In Prop. 15. let the whole line be denominated by $a$, and its greater segment by $x$; then $x^2=a(a-x)$, and $x^2+ax=a^2$: whence $x=\pm\sqrt{\frac{5a^2}{4}}-\frac{a}{2}=\pm a(\sqrt{\frac{5}{4}}-\frac{1}{2})$. Hence, if unit represent the whole line, the greater segment is .61803398428, &c. and the smaller segment .38196601572, &c.

From the corollary an extremely neat approximation is likewise obtained. Assuming the segments of the divided line to be at first equal, and each denoted by 1, the following successive numbers will result from a continued summation:

1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, &c.

which are thus composed,

$1+2=3,\ 2+3=5,\ 3+5=8,\ 5+8=13,\ 8+13=21$, &c.

These numbers form, therefore, the simplest recurring series.

If the original line contained 144 equal parts, its greater segment would include 89, and its smaller segment 55 of these parts, very nearly; but $55\times 144=7920$, being only one less than 7921, the square of 89.

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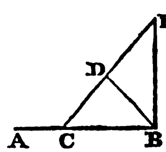
## BOOK III.

1. Proposition eighth. Hence angles are sometimes measured by a circular instrument, from a point in the circumference, as well as from the centre.

2. Proposition ninth. On this proposition depends the construction of amphitheatres; for the visual magnitude of an object is measured by the angle which it subtends at the eye, and consequently the whole extent of the stage, the intermediate objects being purposely darkened or obscured, will be seen with equal advantage by every spectator seated in the same arc of a circle.

3. Proposition eleventh. To *erect* a perpendicular, any point D is taken, as in Prop. 5. Book I., and from it a circle is described passing through C and B; the diameter CDF, by its intersection at the point B, determines the position of the perpendicular BF. To *let fall* a perpendicular, draw to AB any straight line FC, which bisect in D, and from this point as a centre describe a circle through the points C, B and F; FB is the perpendicular required.



## BOOK IV.

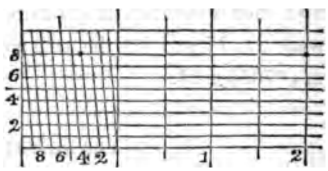
1. The equilateral triangle, the square, the pentagon, the hexagon, and other polygons derived from these, were the only regular figures known to the Greeks. The inscription of all the rest has for ages been supposed absolutely to transcend the powers of Elementary Geometry. But a curious and most unexpected discovery was lately made by Mr Gauss, Professor of Astronomy in the University of Göttingen, who has demonstrated, in a work entitled *Disquisitiones Arithmeticæ*, published at Brunswick in 1801, that certain very complex polygons can yet be described merely by the help of circles. Thus, a regular polygon containing 17, 257, 65537, &c. sides, is capable of being thus inscribed; and in general, when the number of sides may be denoted by $2^n+1$, and is at the same time a prime number.

## BOOK V.

1. Proposition fourth. It will be proper here to notice the several methods adopted in practice, for the minute subdivision of lines. The earliest of these—the *diagonal scale*—depending immediately on the proposition in the text, is of the most extensive use, and constituted the first improvement of astronomical instruments.

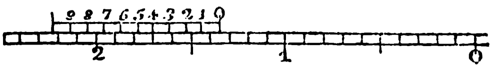
Thus, in the figure annexed, the extreme portion of the horizontal line is divided into ten equal parts, each of which again is virtually subdivided into ten secondary parts. The subdivision is effected by means of diagonal lines, which decline from the perpen-

dicular by intervals equal to the primary divisions, and which are cut transversely into ten equal segments by equidistant parallels. Suppose, for example, it were required to find the length of 2 and 38—100 parts of a division; place one foot of the compasses in the second vertical line at the eighth interval which is marked with a dot, and extend the other foot, along the parallel, to the dot on the third diagonal. The distance between these dots may, however, express indifferently 2.38, 23.8, or 238, according to the assumed magnitude of the primary unit.




But the method of subdivision introduced by Vernier is simpler and more ingenious. It is founded on the difference of two approximating scales, one of which is moveable. Thus, if a space equal to $n$—I part on the limb of the instrument be divided into $n$ parts, these evidently will each of them be smaller than the former, by the $n$th part of a division. Wherefore, on shifting forward this attached scale, the quantity of aberration will diminish at each successive division, till a new coincidence obtains, and then the number of those divisions on that scale will mark the fractional value of the displacement.

Thus in the annexed figure, nine divisions of the primary scale, forming ten equal parts on the attached or sliding scale, the moveable zero stands beyond the first interval between the third and fourth division. To find this minute difference, observe where the opposite sections of the scales come to coincide, which occurs under the fourth division of the sliding scale, and therefore indicates the quantity 1.34.




2. Proposition nineteenth. This most useful proposition in practical geometry was known to the Greeks of Alexandria, and by

them communicated to the Arabians, but seems to have been re-invented in Europe about the latter part of the fifteenth century.

Suppose the sides of the general triangle to be 13, 14, and 15; then is the area $= \sqrt{(21.8.7.6)} = \sqrt{7056} = 84$. If the sides were 21, 17 and 10, the area would be the same, for $\sqrt{(24.3.7.14)} = \sqrt{7056} = 84$.

### PLANE TRIGONOMETRY.

1. In the demonstration of Proposition sixth, a reference should have been made to the construction and similarity of the component triangles, proved in Prop. 19. Book V.

2. The art of SURVEYING, in general, consists in determining the boundaries of an extended surface. When performed in the completest manner, it ascertains the positions of all the prominent objects within the scope of observation, measures their mutual distances and relative heights, and consequently defines the various contours which mark the surface. But the land-surveyor seldom aims at such minute and scrupulous accuracy; his main object is to trace expeditiously the chief boundaries, and to compute the superficial contents of each field. In hilly grounds, however, it is not the absolute surface that is measured, but the diminished quantity which would result, had the whole been reduced to a horizontal plane.

Land is surveyed either by means of the chain simply, or by combining this with a theodolite or some other angular instrument. The several fields are divided into large triangles, of which the sides are measured by the chain; and if the exterior boundary happens to be irregular, the perpendicular distance or offset is taken at each bending. The surface of the component triangles is then computed from their sides. In this method the triangles should be chosen as nearly equilateral as possible; for if they be very oblique, the smallest error in the length of their sides will occasion a wide difference in the estimate of the surface.

The Imperial *chain* is 22 yards, or 66 feet in length, and equi-

valent to four *poles;* it is hence the tenth part of a furlong, or the eightieth part of a mile. The chain is divided into a hundred links, each occupying 7.92 inches. An *acre* contains ten square chains or 100,000 square links. A square mile, therefore, includes 640 acres; and this large measure is deemed sufficient, in rude and savage countries, where vast tracts of new land are allotted, by merely running lines north and south, and intersecting these with perpendiculars, at each interval of a mile.

The best method of surveying, if not always the most expeditious, undoubtedly is to cover the ground with a series of connected triangles, planting the theodolite at each angular point, and computing from some base of considerable extent, which has been selected and measured with nice attention. The labour of transporting the instrument might also in many cases be abridged, by observing at any station the bearings at once of several signals. Since angles are measured more accurately than lines, it would be desirable that surveyors generally employed theodolites of a better construction, and trusted less to the aid of the chain.

3. LEVELLING is a delicate and important branch of general surveying. It discovers the difference of altitude from a combined series of observations carried along an irregular surface, the aggregate of the several descents being deducted from that of the ascents. For this purpose, the staves are placed successively along the line of survey, at proper intervals according to the nature of the ground, not exceeding 400 yards, the levelling instrument being always planted nearly in the middle between them, and directed backwards to the first staff, and then forwards to the second. The final result of a series of operations, or the difference of altitude between the extreme stations, is found by collecting severally the measures of the back and fore observations, the excess of the latter above the former indicating the entire quantity of descent.

To facilitate the calculations in levelling, the rods should be marked with feet, divided into tenths and hundredth parts, instead

of inches. In delicate operations likewise, the instrument should have a micrometer adapted to it, for measuring small vertical angles. By help of a powerful level so fitted, much tedious labour indeed might be spared.

The micrometer either marks minutes and their subdivisions, by the motion of a parallel wire in the focus of the telescope; or it consists in the simple insertion of a bar of mother of pearl, by which the angle of a degree is distinguished into 50 equal parts. But an object at the distance of 3438 times its breadth subtends the angle of a minute; or, at the distance of 2865 times the same, it occupies the 50th part of a degree. Hence, the measure of one minute, being multiplied by 3438, will give the distance; and this again multiplied by the number of intercepted minutes will express the elevation or depression of the distant point. The correction for the effect of curvature modified by refraction, it is easily shown, will be found by squaring half the measure in feet of the angle of a minute. When the micrometer scale is used, the distance will be discovered by multiplying the measure corresponding to one part by 2865, and the square of that measure being divided by *six* will assign the correction due to curvature and refraction.

Suppose, for example, a pole of 20 feet placed on a remote eminence subtended an angle of 2.5′ and the bottom appeared elevated 42′ above the horizon. Dividing 20 by 2.5′, gives 8 feet for the measure of a minute. The distance was therefore $8 \times 3438 = 27504$ feet; the height was $8 \times 42 = 336$; and the depression from curvature and refraction was the square of 4, the half of 8, or 16 feet. Whence the whole difference of level must have been $336 + 16 = 352$. The same observations with the bar of mother of pearl would have been 2.1′ and 35′, from which data similar results would be obtained.

The mode of levelling on a grand scale, or determining the heights of distant mountains, will receive illustration from our Trigonometrical Survey. Selecting the largest triangle in the series, which connects the North of England with the Borders of Scotland: The distance of the station on Cross Fell to that on Wisp Hill, is computed at 235018.6 feet, or 44.511 miles, which, reckoning 6094½ feet for the length of a minute near that parallel,

corresponds, on the surface of the globe, to an arc of 38′ 33.7″. Wisp Hill was seen depressed 30′ 48″ from Cross Fell, which again had a depression of 2′ 31″ when viewed from Wisp Hill. The sum of these depressions is 33′ 19″, which, taken from 38′ 33.7″, the measure of the intercepted arc, or the angle at the centre, leaves 5′ 14.7″, for the joint effect of refraction at both stations. The deflection of the visual ray produced by that cause, which the French philosophers estimate in general at .079, had therefore amounted only to .06805, or to very little more than the *fifteenth* part of the intercepted arc. Hence, the true depression of Wisp Hill was 30′ 48″—16′ 39.5″=14′ 8.5″, and consequently, estimating from the given distance, it is 967 feet lower than Cross Fell.

From Wisp Hill, the top of Cheviot appeared exactly on the same level, at the distance of 185023.9 feet, or 35.0424 miles. Wherefore, two-thirds of the square of this last number, or 1819, would express in feet the approximate height of Cheviot above Wisp Hill. But refraction gave the mountain a more towering elevation than it really had; and the measure being reduced in the former ratio of 38′ 33.7″ to 33′ 19″, is hence brought down to 708 feet.

I shall subjoin another example, which affords an approximation to the diameter of our globe. From the station at the Observatory on the Calton-hill, at the altitude of 350 feet, the horizon of the sea was found depressed 18′ 12″. But refraction being supposed to have diminished the effect by one-twelfth part, if the eleventh part be added of this remaining quantity, there will result 19′ 51¼″ for the true measure of depression. The angle at the centre is consequently the half of 19′ 51¼″ or 9′ 55⅝″; wherefore, *sin* 9′ 55⅝″ : R : : 350 : 121.205 feet, or 22.9555 miles, the distance at which the extreme visual ray grazes the sea. Again, *sin* 19′ 51¼″ : R : : 22.9555 : 3975 miles, the radius of the earth, the double of which, or 7950, is a near approximation to the real measure, or 7912.

4. Maritime Surveying is of a mixed nature: It not only determines the positions of the remarkable headlands, and other conspicuous objects that present themselves along the vicinity of a

coast, but likewise ascertains the situation of the various inlets, rocks, shallows and soundings which occur in approaching the shore. To survey a new or inaccessible coast, two boats are moored at a proper interval, which is carefully measured on the surface of the water; and from each boat the bearings of all the prominent points of land are taken by means of an azimuth compass, or the angles subtended by these points and the other boat are measured by a Hadley's sextant. Having now drawn on paper the base to any scale, straight lines radiating from each end at the observed angles will by their intersections give the positions of the several points from which the coast may be sketched.

But a chart is more accurately constructed, by combining a survey made on land, with observations taken on the water. A smooth level piece of ground is chosen, on which a base of considerable length is measured out, and station staves are fixed at its extremities. If no such place can be found, the mutual distance and position of two points conveniently situate for planting the staves, though divided by a broken surface, are determined from one or more triangles, which connect with a shorter and temporary base assumed near the beach. A boat then explores the offing, and at every rock, shallow, or remarkable sounding, the bearings of the station staves are noticed. These observations furnish so many triangles, from which the situation of the several points are easily ascertained. —When a correct map of the coast can be procured, the labour of executing a maritime survey is materially shortened. From each notable point of the surface of the water, the bearings of two known objects on the land are taken, or the intermediate angles subtended by three such objects are observed. In the first case, those various points have their situations ascertained by Prop. 21. and the second case by Prop. 25. of the Trigonometry. To facilitate the last construction, an instrument called the *Station-Pointer* has been invented, consisting of three brass rulers, which open and set at the given angles.

THE END.